

# ON THE GLOBAL SOLUTION OF 3-D MHD SYSTEM WITH INITIAL DATA NEAR EQUILIBRIUM

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**ABSTRACT.** In this paper, we prove the global existence of smooth solutions to the three-dimensional incompressible magneto-hydrodynamical system with initial data close enough to the equilibrium state,  $(e_3, 0)$ . Compared with the previous works [21, 29], here we present a new Lagrangian formulation of the system, which is a damped wave equation and which is non-degenerate only in the direction of the initial magnetic field. Furthermore, we remove the admissible condition on the initial magnetic field, which was required in [21, 29]. By using Frobenius Theorem and anisotropic Littlewood-Paley theory for the Lagrangian formulation of the system, we achieve the global  $L^1$  in time Lipschitz estimate of the velocity field, which allows us to conclude the global existence of solutions to this system. In the case when the initial magnetic field is a constant vector, the large time decay rate of the solution is also obtained.

**Keywords:** Inviscid MHD system, Anisotropic Littlewood-Paley theory, Lagrangian coordinates

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## 1. INTRODUCTION

In this paper, we investigate the global existence of smooth solutions to the following three-dimensional incompressible magnetic hydrodynamical system (or MHD in short) with initial data being sufficiently close to the equilibrium state  $(e_3, 0)$  :

$$(1.1) \quad \begin{cases} \partial_t b + u \cdot \nabla b = b \cdot \nabla u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = b \cdot \nabla b, \\ \operatorname{div} u = \operatorname{div} b = 0, \\ (b, u)|_{t=0} = (b_0, u_0) \quad \text{with} \quad b_0 = e_3 + \varepsilon \phi, \end{cases}$$

where  $b = (b^1, b^2, b^3)$  denotes the magnetic field, and  $u = (u^1, u^2, u^3)$ ,  $p$  the velocity and scalar pressure of the fluid respectively. This MHD system (1.1) with zero diffusivity in the magnetic field equation can be applied to model plasmas when the plasmas are strongly collisional, or the resistivity due to these collisions are extremely small. One may check the references [4, 15, 19] for detailed explanations to this system.

In general, it is not known whether or not classical solutions of (1.1) can develop finite time singularities even in two dimension. In the case when there is full magnetic diffusion in (1.1), Duvaut and Lions [16] established the local existence and uniqueness of solution in the classical Sobolev space  $H^s(\mathbb{R}^d)$ ,  $s \geq d$ , they also proved the global existence of solutions to this system with small initial data; Sermange and Temam [28] proved the global well-posedness of this system in the two space dimension; the first author and Paicu [1] proved similar result as that in [16] for the so-called inhomogeneous MHD system with initial data in the critical spaces. With mixed partial dissipation and additional magnetic diffusion in the two-dimensional MHD system, Cao and Wu [5] (see also [6]) proved that such a system is globally well-posed for any data in  $H^2(\mathbb{R}^2)$ . Very recently, Chemin et al [10] proved the local well-posedness of (1.1) with initial data in the critical Besov

spaces. One may check the survey paper [20] and the references therein for the recent progresses in this direction and also its relations to the incompressible visco-elastic fluid system.

Furthermore, whether there is dissipation or not for the magnetic field of (1.1) is a very important problem also from physics of plasmas. The heating of high temperature plasmas by MHD waves is one of the most interesting and challenging problems of plasma physics especially when the energy is injected into the system at the length scales which are much larger than the dissipative ones. It has been conjectured that in the three-dimensional MHD system, energy is dissipated at a rate that is independent of the ohmic resistivity [12]. In other words, the viscosity (diffusivity) for the magnetic field equation can be zero yet the whole system may still be dissipative. As a first step to investigate this problem, Lin and the second author [22] proved the global well-posedness to a modified three-dimensional MHD system with initial data sufficiently close to the equilibrium state (see [23] for a simplified proof). This problem was partially solved in 2-D by Lin, Xu and the second author in [21] and by Xu and the second author in 3-D in [29] provided that the initial data is near the equilibrium state  $(e_d, 0)$  and the initial magnetic field,  $b_0$ , satisfies the following admissible condition, namely

$$(1.2) \quad \int_{\mathbb{R}} (b_0 - e_3)(Z(t, \alpha)) dt = 0 \quad \text{for all} \quad \alpha \in \mathbb{R}^{d-1} \times \{0\}$$

with  $Z(t, \alpha)$  being determined by

$$\begin{cases} \frac{dZ(t, \alpha)}{dt} = b_0(Z(t, \alpha)), \\ Z(t, \alpha)|_{t=0} = \alpha. \end{cases}$$

In the 2-D case, the restriction (1.2) was removed by Ren, Wu, Xiang and Zhang in [26] by carefully exploiting the divergence structure of the velocity field. Moreover, the authors proved that

$$(1.3) \quad \|\partial_{x_2}^k b(t)\|_{L^2} + \|\partial_{x_2}^k u(t)\|_{L^2} \leq C \langle t \rangle^{-\frac{s+k}{2}} \quad \text{for any} \quad s \in ]0, 1/2[ \quad \text{and} \quad k = 0, 1, 2.$$

A more elementary existence proof was also given by Zhang in [31]. Very recently, Ren, Xiang and Zhang extended this well-posedness result to the strip domain in [27]. The goal of this paper is to remove the assumption (1.2) and improve the decay estimates (1.3) for the limiting case  $s = \frac{1}{2}$  in three space dimension.

Before we present the function spaces we are going to work with in this context, let us briefly recall some basic facts on Littlewood-Paley theory (see e.g. [2]). Let  $\varphi$  and  $\chi$  be smooth functions supported in  $\mathcal{C} \stackrel{\text{def}}{=} \{\tau \in \mathbb{R}^+, \frac{3}{4} \leq \tau \leq \frac{8}{3}\}$  and  $\mathfrak{B} \stackrel{\text{def}}{=} \{\tau \in \mathbb{R}^+, \tau \leq \frac{4}{3}\}$  respectively such that

$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\tau) = 1 \quad \text{for} \quad \tau > 0 \quad \text{and} \quad \chi(\tau) + \sum_{j \geq 0} \varphi(2^{-j}\tau) = 1 \quad \text{for} \quad \tau \geq 0.$$

For  $a \in \mathcal{S}'(\mathbb{R}^3)$ , we set

$$(1.4) \quad \begin{aligned} \Delta_k^h a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-k}|\xi_h|)\widehat{a}), & S_k^h a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-k}|\xi_h|)\widehat{a}), \\ \Delta_\ell^v a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-\ell}|\xi_3|)\widehat{a}), & S_\ell^v a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-\ell}|\xi_3|)\widehat{a}), \quad \text{and} \\ \Delta_j a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\varphi(2^{-j}|\xi|)\widehat{a}), & S_j a &\stackrel{\text{def}}{=} \mathcal{F}^{-1}(\chi(2^{-j}|\xi|)\widehat{a}), \end{aligned}$$

where  $\xi_h = (\xi_1, \xi_2)$ ,  $\xi = (\xi_h, \xi_3)$ ,  $\mathcal{F}a$  and  $\widehat{a}$  denote the Fourier transform of the distribution  $a$ . The dyadic operators satisfy the property of almost orthogonality:

$$(1.5) \quad \Delta_k \Delta_j a \equiv 0 \quad \text{if} \quad |k - j| \geq 2 \quad \text{and} \quad \Delta_k (S_{j-1} a \Delta_j b) \equiv 0 \quad \text{if} \quad |k - j| \geq 5.$$

Similar properties hold for  $\Delta_k^h$  and  $\Delta_\ell^v$ .

**Definition 1.1** (Definition 2.15 of [2]). Let  $(p, r) \in [1, +\infty]^2$ ,  $s \in \mathbb{R}$  and  $a \in \mathcal{S}'_h(\mathbb{R}^3)$ , which means  $a \in \mathcal{S}'(\mathbb{R}^d)$  and  $\lim_{j \rightarrow -\infty} \|\chi(2^{-j}D)a\|_{L^\infty} = 0$ , we set

$$\|a\|_{\dot{B}_{p,r}^s} \stackrel{\text{def}}{=} \left(2^{js} \|\Delta_j a\|_{L^p}\right)_{\ell^r}.$$

- For  $s < \frac{3}{p}$  (or  $s = \frac{3}{p}$  if  $r = 1$ ), we define  $\dot{B}_{p,r}^s(\mathbb{R}^3) \stackrel{\text{def}}{=} \{a \in \mathcal{S}'_h(\mathbb{R}^3) \mid \|a\|_{\dot{B}_{p,r}^s} < \infty\}$ .
- If  $k \in \mathbb{N}$  and  $\frac{3}{p} + k - 1 \leq s < \frac{3}{p} + k$  (or  $s = \frac{3}{p} + k$  if  $r = 1$ ), then  $\dot{B}_{p,r}^s(\mathbb{R}^3)$  is defined as the subset of distributions  $a \in \mathcal{S}'_h(\mathbb{R}^3)$  such that  $\partial^\beta a \in \dot{B}_{p,r}^{s-k}(\mathbb{R}^3)$  whenever  $|\beta| = k$ .

When  $p = 2$  and  $r = 1$ , we denote  $\dot{B}_{2,1}^s$  by  $\dot{B}^s$  and  $\dot{B}^s(\mathbb{R}_{x_h}^2)$  by  $\dot{B}_h^s$ .

Due to the anisotropic spectral properties of the linearized equation to (1.1) (see Section 3 for more explanation), we need also the following anisotropic type Besov norm from [22, 21]:

**Definition 1.2.** Let  $s_1, s_2 \in \mathbb{R}$ ,  $r_1, r_2 \in [1, \infty]$  and  $a \in \mathcal{S}'_h(\mathbb{R}^3)$ , we define the norm

$$\|a\|_{\mathcal{B}_{r_1,r_2}^{s_1,s_2}} \stackrel{\text{def}}{=} \left(2^{js_1} (2^{\ell s_2} \|\Delta_j \Delta_\ell^\vee a\|_{L^2})_{\ell^{r_2}}\right)_{\ell^{r_1}}.$$

In particular, when  $r_1 = r_2 = 1$ , we denote  $\|a\|_{\mathcal{B}^{s_1,s_2}} \stackrel{\text{def}}{=} \|a\|_{\mathcal{B}_{1,1}^{s_1,s_2}}$

The main result of this paper is as follows:

**Theorem 1.1.** Let  $e_3 = (0, 0, 1)$ ,  $b_0 = e_3 + \varepsilon \phi$  with  $\phi = (\phi_1, \phi_2, \phi_3) \in C_c^3(\mathbb{R}^3)$  and  $\text{div } \phi = 0$ , let  $u_0 \in H^s(\mathbb{R}^3)$  for  $s \in ]3/2, 3]$ . Then there exist sufficiently small positive constants  $\varepsilon_0, c_0$  such that if

$$(1.6) \quad \|u_0\|_{\dot{B}^{\frac{1}{2}}} \leq c_0 \quad \text{and} \quad \varepsilon \leq \varepsilon_0,$$

(1.1) has a unique global solution  $(b, u)$  so that for any  $T > 0$ ,  $b - e_3 \in C([0, T]; H^s(\mathbb{R}^3))$ ,  $u \in C([0, T]; H^s(\mathbb{R}^3))$  with  $\nabla u \in L^2([0, T]; H^s(\mathbb{R}^3))$  and  $\nabla p \in C([0, T]; H^{s-1}(\mathbb{R}^3))$ . Moreover, in the case when  $\varepsilon = 0$ , and under the additional assumption that

$$(1.7) \quad \|u_0\|_{\mathcal{B}^{0,0}} + \|u_0\|_{\mathcal{B}^{3,0}} + \|u_0\|_{\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}}} + \|u_0\|_{\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}}} \leq \delta_0,$$

for some  $\delta_0$  sufficiently small, one has

$$(1.8) \quad \|u(t)\|_{H^2} + \|b(t) - e_3\|_{H^2} \leq C \langle t \rangle^{-\frac{1}{4}} \quad \text{with} \quad \langle t \rangle = (1 + t^2)^{\frac{1}{2}}.$$

**Remark 1.1.** (1) Our approach to prove Theorem 1.1 works in both three space dimension and two space dimension. Moreover, for a concise presentation, here we did not optimize the regularity of the initial magnetic field.

(2) In general, it is impossible to propagate the anisotropic regularities for the solutions of hyperbolic systems (it is only possible for conormal regularities (see [7] for instance)). Since we need to use the anisotropic regularities of the solution in order to prove the decay estimate (1.8), we are forced to study the large time behavior of the solutions to the Lagrangian formulation of (1.1).

(3) It is easy to observe from (2.12), the equivalent Lagrangian formulation of (1.1), that, the solution  $(b - e_3, u)$  to (1.1) does not decay to zero as time goes to  $\infty$  when the initial magnetic field is not a constant vector. That is the reason why we only investigate the large time behavior of the solution to (1.1) when  $b_0 = e_3$ .

(4) More detailed decay estimates of the solution in the Lagrangian coordinate will be presented in Theorem 2.1 of Section 2.

Let us complete this section by the notations we shall use in this context.

**Notation.** For any  $s \in \mathbb{R}$ , we denote by  $H^s(\mathbb{R}^3)$  the classical  $L^2$  based Sobolev spaces with the norm  $\|\cdot\|_{H^s}$ , while  $\dot{H}^s(\mathbb{R}^3)$  the classical homogenous Sobolev spaces with the norm  $\|\cdot\|_{\dot{H}^s}$ . For

$a \lesssim b$ , we mean that there is a uniform constant  $C$ , which may be different on different lines, such that  $a \leq Cb$ , and  $a \sim b$  means that both  $a \lesssim b$  and  $b \lesssim a$ . We shall denote by  $(a|b)$  the  $L^2(\mathbb{R}^3)$  inner product of  $a$  and  $b$ .  $(d_{j,k})_{j,k \in \mathbb{Z}}$  (resp.  $(d_j)_{j \in \mathbb{Z}}$ ) will be a generic element of  $\ell^1(\mathbb{Z}^2)$  (resp.  $\ell^1(\mathbb{Z})$ ) so that  $\sum_{j,k \in \mathbb{Z}} d_{j,k} = 1$  (resp.  $\sum_{j \in \mathbb{Z}} d_j = 1$ ). Finally, we denote by  $L_T^p(L_h^q(L_v^r))$  the space  $L^p([0, T]; L^q(\mathbb{R}_{x_h}^2; L^r(\mathbb{R}_{x_3})))$  with  $x_h = (x_1, x_2)$ .

## 2. LAGRANGIAN FORMULATION OF (1.1)

In view of Proposition 6.1 of [29] (see also Proposition 7.1 below), the main difficulty to prove the existence part of Theorem 1.1 is to achieve the  $L^1(\mathbb{R}^+; \text{Lip}(\mathbb{R}^3))$  estimate of the velocity field to the appropriate approximate solutions of (1.1). Due to the difficulty mentioned in (2) of Remark 1.1, we used Lagrangian formulation of (1.1) in the previous works [21, 29].

Let us now explain the main idea for the Lagrangian formulation of (1.1) in [21, 29]. Taking the 3-D case for example, given  $b_0$  satisfying the admissible condition (1.2), the authors first construct a matrix  $U_0 = (\bar{b}_0, \tilde{b}_0, b_0)$  with  $\bar{b}_0 = (\bar{b}_0^1, \bar{b}_0^2, \bar{b}_0^3)^t$  and  $\tilde{b}_0 = (\tilde{b}_0^1, \tilde{b}_0^2, \tilde{b}_0^3)^t$ , so that there hold

$$(2.1) \quad \det U_0 = 1, \quad \text{div } \bar{b}_0 = 0 \quad \text{and} \quad \text{div } \tilde{b}_0 = 0.$$

Then instead of solving (1.1), the authors proposed to solve

$$(2.2) \quad \begin{cases} \partial_t U + u \cdot \nabla U = \nabla u U, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = b \cdot \nabla b, \\ \text{div } u = 0 \quad \text{and} \quad \text{div } U = 0, \\ U|_{t=0} = U_0, \quad u|_{t=0} = u_0. \end{cases}$$

Motivated by the Lagrangian formulation of the visco-elastic system in [30], the authors gave the following Lagrangian formulation of the System (2.2):

$$(2.3) \quad \begin{cases} Y_{tt} - \Delta_y Y_t - \partial_{y_3}^2 Y = (\nabla_Y \cdot \nabla_Y - \Delta_y) Y_t - \nabla_Y q, \\ \nabla_y \cdot Y = \nabla_y \cdot Y_0 - \int_0^t (\nabla_Y - \nabla_y) \cdot Y_s ds, \\ Y|_{t=0} = Y_0, \quad Y_t|_{t=0} = Y_1, \end{cases}$$

for  $Y$  and  $\nabla_Y$  being determined by (2.11) below. It is the restriction (2.1) that requires the admissible condition (1.2).

Here we shall give a more direct Lagrangian formulation of the System (1.1), which will be based on Lemma 1.4 of [24]. In order to do so, let us first give an equivalent formulation of (1.1), which does not involve the pressure function. Indeed we get, by taking the space divergence to the velocity equation of (1.1), that

$$(2.4) \quad \Delta p = \text{div div}(b \otimes b - u \otimes u), \quad \text{or} \quad p \stackrel{\text{def}}{=} \Delta^{-1} \text{div div}(b \otimes b - u \otimes u).$$

Then (1.1) can be equivalently reformulated as

$$(2.5) \quad \begin{cases} \partial_t b + u \cdot \nabla b = b \cdot \nabla u, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t u + u \cdot \nabla u - \Delta u + \nabla p = b \cdot \nabla b, \\ (b, u)|_{t=0} = (b_0, u_0), \quad \text{and} \quad \text{div } b_0 = \text{div } u_0 = 0, \end{cases}$$

with  $p$  given by (2.4). And then just as in Chapter 1 of [8] for the incompressible Euler system, the divergence free condition of  $u$  and  $b$  can be derived by the initial condition  $\text{div } b_0 = \text{div } u_0 = 0$  and the evolution equation of  $\text{div } b$  and  $\text{div } u$ .

Now let  $(b, u)$  be a smooth enough solution of (2.5), we define the Lagrangian trajectory  $X(t, y)$  by

$$(2.6) \quad \begin{cases} \frac{d}{dt} X(t, y) = u(t, X(t, y)), \\ X(0, y) = y, \end{cases}$$

which yields for  $i, j \in \{1, 2, 3\}$  that

$$\det\left(\frac{\partial X}{\partial y}\right) = 1 \quad \text{and} \quad \frac{d}{dt} \frac{\partial X^i(t, y)}{\partial y_j} = \frac{\partial u^i}{\partial x_\ell}(t, X(t, y)) \frac{\partial X^\ell(t, y)}{\partial y_j},$$

and

$$(2.7) \quad \frac{d}{dt} \left( b_0^j(y) \frac{\partial X^i(t, y)}{\partial y_j} \right) = \left( b_0^j(y) \frac{\partial X^\ell(t, y)}{\partial y_j} \right) \frac{\partial u^i}{\partial x_\ell}(t, X(t, y)).$$

On the other hand, it follows from the magnetic field equation of (2.5) and (2.6) that

$$\frac{db^i(t, X(t, y))}{dt} = b(t, X(t, y)) \cdot \nabla u^i(t, X(t, y)),$$

which together with (2.7) ensures that

$$(2.8) \quad b^i(t, X(t, y)) = b_0^j(y) \frac{\partial X^i(t, y)}{\partial y_j} = b_0(y) \cdot \nabla_y X^i(t, y) \stackrel{\text{def}}{=} \partial_{b_0} X^i(t, y).$$

For any smooth function  $f$ , we deduce from chain rule that

$$\frac{\partial f(X(t, y))}{\partial y_j} = \left( \frac{\partial f}{\partial x_\ell} \right)(X(t, y)) \frac{\partial X^\ell(t, y)}{\partial y_j}.$$

Let us denote the inverse matrix of  $\frac{\partial X(t, y)}{\partial y}$  by  $\mathcal{A}(t, y) = (a_{ij}(t, y))$ . Then we have

$$(2.9) \quad \left( \frac{\partial f}{\partial x_i} \right)(X(t, y)) = a_{ji}(t, y) \frac{\partial f(X(t, y))}{\partial y_j} \quad \text{or} \quad (\nabla_x f)(X(t, y)) = \mathcal{A}^t \nabla_y (f(X(t, y))).$$

By virtue of (2.8) and (2.9), we infer

$$(2.10) \quad \begin{aligned} (b^j \partial_j b^i)(t, X(t, y)) &= b_0^k(y) \frac{\partial X^j(t, y)}{\partial y_k} a_{\ell j}(t, y) \frac{\partial b^i(t, X(t, y))}{\partial y_\ell} \\ &= b_0^k(y) \delta_{k\ell} \partial_{y_\ell} (\partial_{b_0} X^i(t, y)) \\ &= \partial_{b_0}^2 X^i(t, y). \end{aligned}$$

Let us denote

$$(2.11) \quad \begin{aligned} X(t, y) &= y + \int_0^t u(t', X(t', y)) dt' \stackrel{\text{def}}{=} y + Y(t, y), \quad \mathbf{u}(t, y) \stackrel{\text{def}}{=} u(t, X(t, y)), \\ \mathbf{b}(t, y) &\stackrel{\text{def}}{=} b(t, X(t, y)), \quad \mathbf{p}(t, y) \stackrel{\text{def}}{=} p(t, X(t, y)), \quad \mathcal{A} \stackrel{\text{def}}{=} (Id + \nabla_y Y)^{-1} \quad \text{and} \quad \nabla_Y \stackrel{\text{def}}{=} \mathcal{A}^t \nabla_y. \end{aligned}$$

Then thanks to (2.5), (2.8) and (2.10), we write

$$(2.12) \quad \begin{cases} \mathbf{b}(t, y) = \partial_{b_0} X(t, y), & \nabla_Y \cdot \mathbf{b} = 0, \\ Y_{tt} - \Delta_y Y_t - \partial_{b_0}^2 Y = \partial_{b_0} b_0 + g, \\ Y|_{t=0} = Y_0 = 0, & Y_t|_{t=0} = Y_1 = u_0(y), \end{cases}$$

where

$$(2.13) \quad \begin{aligned} g &= \text{div}_y [(\mathcal{A} \mathcal{A}^t - Id) \nabla_y Y_t] - \mathcal{A}^t \nabla_y \mathbf{p}, \quad \partial_{b_0} \stackrel{\text{def}}{=} b_0 \cdot \nabla_y, \quad \text{and} \\ (\Delta_x p)(t, X(t, y)) &= \sum_{i,j=1}^3 \nabla_{Y^i} \nabla_{Y^j} (\partial_{b_0} X^i \partial_{b_0} X^j - Y_t^i Y_t^j)(t, y). \end{aligned}$$

Compared with the Lagrangian formulation (2.3) in [21, 29],  $\partial_{y_3}^2 Y$  there is now replaced by  $\partial_{b_0}^2 Y$ , which causes new difficulty of the variable coefficients for the linearized system.

In what follows, we assume that

$$(2.14) \quad \text{supp}(b_0(x_h, \cdot) - e_3) \subset [0, K] \quad \text{and} \quad b_0^3 \neq 0.$$

Due to the difficulty of the variable coefficients for the linearized system of (2.12), we shall use Frobenius Theorem type argument to find a new coordinate system  $\{z\}$  so that  $\partial_{b_0} = \partial_{z_3}$ . Then we can use anisotropic Littlewood-Paley analysis to achieve the  $L_t^1(\text{Lip})$  estimate for  $Y_t$ . Toward this, let us define

$$(2.15) \quad \begin{cases} \frac{dy_1}{dy_3} = \frac{b_0^1}{b_0^3}(y_1, y_2, y_3), & y_1|_{y_3=0} = w_1, \\ \frac{dy_2}{dy_3} = \frac{b_0^2}{b_0^3}(y_1, y_2, y_3), & y_2|_{y_3=0} = w_2, \\ y_3 = w_3, \end{cases}$$

and

$$(2.16) \quad z_1 = w_1, \quad z_2 = w_2, \quad z_3 = w_3 + \int_0^{w_3} \left( \frac{1}{b_0^3(y(w))} - 1 \right) dw'_3.$$

Then we have

$$(2.17) \quad \begin{aligned} \partial_{b_0} f(y) &= b_0^3 \left( \frac{\partial y_1}{\partial w_3} \frac{\partial f(y)}{\partial y_1} + \frac{\partial y_2}{\partial w_3} \frac{\partial f(y)}{\partial y_2} + \frac{\partial f(y)}{\partial w_3} \right) \\ &= b_0^3(y(w)) \frac{\partial f(y(w))}{\partial w_3} = \frac{\partial f(y(w(z)))}{\partial z_3}, \quad \text{and} \\ \partial_{z_i}(f(y(w(z)))) &= \frac{\partial f}{\partial y_j}(y(w(z))) \frac{\partial y_j(w(z))}{\partial z_i} \quad \text{or} \\ \nabla_y &= \nabla_Z = \mathcal{B}^t(z) \nabla_z \quad \text{with} \quad \mathcal{B}(z) = \left( \frac{\partial y(w(z))}{\partial z} \right)^{-1}. \end{aligned}$$

It is easy to observe that

$$\begin{aligned} \mathcal{B}(z) &= \left( \frac{\partial y(w(z))}{\partial z} \right)^{-1} = \left( \frac{\partial y(w(z))}{\partial w} \times \frac{\partial w(z)}{\partial z} \right)^{-1} \\ &= \left( \frac{\partial w(z)}{\partial z} \right)^{-1} \left( \frac{\partial y(w(z))}{\partial w} \right)^{-1} = \left( \frac{\partial z}{\partial w} \right) \left( \frac{\partial y(w(z))}{\partial w} \right)^{-1}. \end{aligned}$$

Yet it follows from (2.15) that

$$(2.18) \quad \begin{aligned} \left( \frac{\partial y(w)}{\partial w} \right) &= \begin{pmatrix} 1 & 0 & \frac{b_0^1}{b_0^3} \\ 0 & 1 & \frac{b_0^2}{b_0^3} \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} \int_0^{w_3} \frac{\partial}{\partial y_1} \left( \frac{b_0^1}{b_0^3} \right) dy'_3 & \int_0^{w_3} \frac{\partial}{\partial y_2} \left( \frac{b_0^1}{b_0^3} \right) dy'_3 & 0 \\ \int_0^{w_3} \frac{\partial}{\partial y_1} \left( \frac{b_0^2}{b_0^3} \right) dy'_3 & \int_0^{w_3} \frac{\partial}{\partial y_2} \left( \frac{b_0^2}{b_0^3} \right) dy'_3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial w_1} & \frac{\partial y_1}{\partial w_2} & \frac{\partial y_1}{\partial w_3} \\ \frac{\partial y_2}{\partial w_1} & \frac{\partial y_2}{\partial w_2} & \frac{\partial y_2}{\partial w_3} \\ \frac{\partial y_3}{\partial w_1} & \frac{\partial y_3}{\partial w_2} & \frac{\partial y_3}{\partial w_3} \end{pmatrix} \\ &\stackrel{\text{def}}{=} A_1(y(w)) + A_2(y(w)) \left( \frac{\partial y(w)}{\partial w} \right), \end{aligned}$$

which gives

$$(2.19) \quad \left( \frac{\partial y(w)}{\partial w} \right) = (Id - A_2(y(w)))^{-1} A_1(y(w)).$$

While it is easy to observe that

$$(2.20) \quad \left( \frac{\partial z(w)}{\partial w} \right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \int_0^{w_3} \frac{\partial}{\partial w_1} \left( \frac{1}{b_0^3(y(w))} \right) dw'_3 & \int_0^{w_3} \frac{\partial}{\partial w_2} \left( \frac{1}{b_0^3(y(w))} \right) dw'_3 & \frac{1}{b_0^3} \end{pmatrix} \stackrel{\text{def}}{=} A_3(w).$$

As a consequence, we obtain

$$(2.21) \quad \begin{aligned} y(w) &= (y_h(w_h, w_3), w_3), \quad w(z) = (z_h, w_3(z)), \quad \text{and} \quad y(w(z)) = (y_h(z_h, w_3(z)), w_3(z)), \\ \mathcal{B}(z) &= A_3(w(z)) A_1^{-1}(y(w(z))) (Id - A_2(y(w(z)))) \end{aligned}$$

with the matrices  $A_1, A_2, A_3$  being given by (2.18) and (2.20) respectively.

For simplicity, let us abuse the notation that  $Y(t, z) = Y(t, y(w(z)))$ . Then the System (2.12) becomes

$$(2.22) \quad \begin{cases} Y_{tt} - \Delta_z Y_t - \partial_{z_3}^2 Y = (\nabla_Z \cdot \nabla_Z - \Delta_z) Y_t + \partial_{z_3} b_0(y(w(z))) + g(y(w(z))), \\ Y|_{t=0} = Y_0 = 0, \quad Y_t|_{t=0} = Y_1(z) = u_0(y(w(z))), \end{cases}$$

for  $g$  given by (2.12). Since  $\partial_{z_3} b_0(y(w(z)))$  in the source term is a time independent function, we now introduce a correction term  $\tilde{Y}$  so that  $Y = \tilde{Y} + \bar{Y}$  and

$$(2.23) \quad \partial_{z_3} \tilde{Y}(z) = e_3 - b_0(y(w(z))).$$

Then

$$\partial_{z_3} (\partial_{z_3} \tilde{Y} + b_0(y(w(z)))) = 0,$$

and  $\bar{Y}$  solves

$$(2.24) \quad \begin{cases} \bar{Y}_{tt} - \Delta_z \bar{Y}_t - \partial_{z_3}^2 \bar{Y} = f, \\ \bar{Y}|_{t=0} = \bar{Y}_0 = -Y, \quad \bar{Y}_t|_{t=0} = Y_1, \end{cases}$$

with

$$(2.25) \quad \begin{aligned} \mathcal{A} &= \left( Id + \mathcal{B}^t \nabla_z \tilde{Y} + \mathcal{B}^t \nabla_z \bar{Y} \right)^{-1}, \quad \text{and} \\ f &= \mathcal{B}^t \nabla_z \cdot [(\mathcal{A} \mathcal{A}^t - Id) \mathcal{B}^t \nabla_z \bar{Y}_t] + \mathcal{B}^t \nabla_z \cdot (\mathcal{B}^t \nabla_z \bar{Y}_t) - \Delta_z \bar{Y}_t - (\mathcal{B} \mathcal{A})^t \nabla_z \mathbf{p}. \end{aligned}$$

In order to handle the term  $\nabla_z \mathbf{p}$  in the source term  $f$ , we need the following lemma:

**Lemma 2.1.** *Let  $X(y)$  be a  $C^1$  diffeomorphism over  $\mathbb{R}^3$  and  $H$  be a  $C^1$  vector field. Then one has*

$$(2.26) \quad (\operatorname{div}_x H)(X(y)) = \det\left(\frac{\partial X}{\partial y}\right)^{-1} \operatorname{div}_y \left( \det\left(\frac{\partial X}{\partial y}\right) \left(\frac{\partial X}{\partial y}\right)^{-1} H(X(y)) \right).$$

*Proof.* The proof of this lemma basically follows from that of Lemma A.1 in [14], where the authors proved (2.26) for the case when  $\det(\frac{\partial X}{\partial y}) = 1$ . Let  $\psi$  be a test function, we denote  $\bar{\psi}(y) \stackrel{\text{def}}{=} \psi(X(y))$ . Then in view of (2.9), one has

$$\begin{aligned} \int_{\mathbb{R}^3} \bar{\psi}(y) (\operatorname{div}_x H)(X(y)) dy &= \int_{\mathbb{R}^3} \psi(x) (\operatorname{div}_x H)(x) \det\left(\frac{\partial X}{\partial y}\right)^{-1} dx \\ &= - \int_{\mathbb{R}^3} \nabla_x \left( \psi(x) \det\left(\frac{\partial X}{\partial y}\right)^{-1} \right) \cdot H(x) dx \\ &= - \int_{\mathbb{R}^3} \nabla_y \left( \bar{\psi}(y) \det\left(\frac{\partial X}{\partial y}\right)^{-1} \right) \left( \frac{\partial X}{\partial y} \right)^{-1} \bar{H}(y) \det\left(\frac{\partial X}{\partial y}\right) dy \\ &= \int_{\mathbb{R}^3} \bar{\psi}(y) \det\left(\frac{\partial X}{\partial y}\right)^{-1} \operatorname{div}_y \left( \left( \frac{\partial X}{\partial y} \right)^{-1} \bar{H}(y) \det\left(\frac{\partial X}{\partial y}\right) \right) dy \end{aligned}$$

This leads to (2.26). □

In particular, if  $\det(\frac{\partial X}{\partial y}) = 1$ , one has

$$(2.27) \quad (\operatorname{div}_x H)(X(y)) = \operatorname{div}_y \left( \left( \frac{\partial X}{\partial y} \right)^{-1} H(X(y)) \right),$$

which recovers Lemma A.1 in [14].

Let us now turn to the calculation of the pressure function in the Lagrangian coordinate. We denote  $\mathcal{Y}(t, y) \stackrel{\text{def}}{=} Y(t, y) - \tilde{\mathcal{Y}}(y)$  with  $\tilde{\mathcal{Y}}(y)$  being determined by

$$(2.28) \quad \partial_{b_0} \tilde{\mathcal{Y}}(y) = e_3 - b_0(y).$$

Then in view of (2.11) and (2.28), we infer

$$\begin{aligned}
\sum_{i,j=1}^3 \nabla_{Y^i} \nabla_{Y^j} (\partial_{b_0} X^i \partial_{b_0} X^j) &= \sum_{i,j=1}^3 \nabla_{Y^i} \nabla_{Y^j} ((b_0^i + \partial_{b_0} Y^i)(b_0^j + \partial_{b_0} Y^j)) \\
&= \sum_{i,j=1}^2 \nabla_{Y^i} \nabla_{Y^j} (\partial_{b_0} \mathcal{Y}^i \partial_{b_0} \mathcal{Y}^j) + \nabla_{Y^3}^2 (1 + \partial_{b_0} \mathcal{Y}^3)^2 \\
&\quad + 2 \sum_{i=1}^2 \nabla_{Y^3} \nabla_{Y^i} (\partial_{b_0} \mathcal{Y}^i (1 + \partial_{b_0} \mathcal{Y}^3)).
\end{aligned}$$

However note that  $\nabla_Y \cdot \mathbf{b} = 0$  and (2.28), one has

$$\begin{aligned}
2\nabla_{Y^3}^2 \partial_{b_0} \mathcal{Y}^3 + 2 \sum_{i=1}^2 \nabla_{Y^3} \nabla_{Y^i} \partial_{b_0} \mathcal{Y}^i &= 2\nabla_{Y^3} \sum_{i=1}^3 \nabla_{Y^i} \partial_{b_0} \mathcal{Y}^i \\
&= 2\nabla_{Y^3} \sum_{i=1}^3 \nabla_{Y^i} \partial_{b_0} (X^i - y^i - \tilde{\mathcal{Y}}^i) \\
&= 2\nabla_{Y^3} \nabla_Y \cdot \mathbf{b} \\
&= 0,
\end{aligned}$$

which leads to

$$\sum_{i,j=1}^3 \nabla_{Y^i} \nabla_{Y^j} (\partial_{b_0} X^i \partial_{b_0} X^j) = \sum_{i,j=1}^3 \nabla_{Y^i} \nabla_{Y^j} (\partial_{b_0} \mathcal{Y}^i \partial_{b_0} \mathcal{Y}^j).$$

As a consequence, we deduce from (2.13) and (2.27) that

$$(2.29) \quad \operatorname{div}_y (\mathcal{A} \mathcal{A}^t \nabla_y \mathbf{p}) = \operatorname{div}_y (\mathcal{A} \operatorname{div}_y (\mathcal{A} (\partial_{b_0} \mathcal{Y} \otimes \partial_{b_0} \mathcal{Y} - \mathcal{Y}_t \otimes \mathcal{Y}_t))).$$

On the other hand, it follows (2.28) that  $\tilde{\mathcal{Y}}(y(w(z)))$  solves (2.23). Let us fix  $\tilde{Y}(z) = \tilde{\mathcal{Y}}(y(w(z)))$ . Then we find

$$\mathcal{Y}(y(w(z))) = Y(t, y(w(z))) - \tilde{\mathcal{Y}}(y(w(z))) = Y(t, y(w(z))) - \tilde{Y}(z) = \bar{Y}(t, z).$$

Hence applying (2.26) to (2.29) gives rise to

$$\operatorname{div}_z (\det(\mathcal{B}^{-1}) \mathcal{B} \mathcal{A} \mathcal{A}^t \mathcal{B}^t \nabla_z \mathbf{p}) = \operatorname{div}_z (\mathcal{B} \operatorname{div}_z (\det(\mathcal{B}^{-1}) \mathcal{B} \mathcal{A} (\partial_3 \bar{Y} \otimes \partial_3 \bar{Y} - \bar{Y}_t \otimes \bar{Y}_t))).$$

This yields

$$\begin{aligned}
(2.30) \quad \nabla_z \mathbf{p} &= -\nabla_z \Delta_z^{-1} \operatorname{div}_z (\det(\mathcal{B}^{-1}) (\mathcal{B} \mathcal{A} \mathcal{A}^t \mathcal{B}^t - Id) \nabla_z \mathbf{p}) \\
&\quad -\nabla_z \Delta_z^{-1} \operatorname{div}_z ((\det(\mathcal{B}^{-1}) Id - Id) \nabla_z \mathbf{p}) \\
&\quad + \nabla_z \Delta_z^{-1} \operatorname{div}_z (\mathcal{B} \operatorname{div}_z (\det(\mathcal{B}^{-1}) \mathcal{B} \mathcal{A} (\partial_3 \bar{Y} \otimes \partial_3 \bar{Y} - \bar{Y}_t \otimes \bar{Y}_t))).
\end{aligned}$$

The local well-posedness of the System (1.1) implies the local well-posedness of the System (2.12) and thus the System (2.24). In what follows, we shall only use the System (2.24) to derive the  $L^1(\mathbb{R}^+; \operatorname{Lip}(\mathbb{R}^3))$  estimate for the velocity field  $u$  of (1.1) provided that there holds (1.6).

To restrict the length of this paper, we shall present the details concerning the propagation of regularities of  $Y$  and  $Y_t$  only in the case when  $b_0 = e_3$ , (the general result can be done by the same strategy), which will be enough for us to investigate the decay estimate (1.8). In this case,  $\mathcal{B} = Id$ ,  $\tilde{Y} = 0$ , and (2.12) becomes

$$(2.31) \quad \begin{cases} Y_{tt} - \Delta_y Y_t - \partial_{y_3}^2 Y = f, \\ Y|_{t=0} = Y_0 = 0, \quad Y_t|_{t=0} = Y_1, \end{cases}$$



with

$$(2.32) \quad \begin{aligned} \mathcal{A} &= (Id + \nabla_y Y)^{-1}, \quad f = \nabla_y \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla_y Y_t) - \mathcal{A}^t \nabla_y \mathbf{p}, \quad \text{and} \\ \mathbf{p} &= -\Delta_y^{-1} \operatorname{div}_y ((\mathcal{A}\mathcal{A}^t - Id)\nabla_y \mathbf{p}) + \Delta_y^{-1} \operatorname{div}_y (\mathcal{A} \operatorname{div}_y (\mathcal{A}(\partial_{y_3} Y \otimes \partial_{y_3} Y - Y_t \otimes Y_t))). \end{aligned}$$

The main result concerning the propagation of regularities and the large time decay estimate for the solutions of (2.31) is listed as follows:

**Theorem 2.1.** *Let  $Y_0 \in \mathcal{B}^{2,0} \cap \mathcal{B}^{5,0} \cap \mathcal{B}^{0,1} \cap \mathcal{B}^{3,1}$  and  $Y_1 \in \mathcal{B}^{0,0} \cap \mathcal{B}^{3,0}$ . Then under the assumption that*

$$(2.33) \quad \mathbf{g}(0) \leq c_0 \quad \text{with} \quad \mathbf{g}(s) \stackrel{\text{def}}{=} \|\partial_3 Y_0\|_{\mathcal{B}^{s,0}} + \|Y_0\|_{\mathcal{B}^{s+2,0}} + \|Y_1\|_{\mathcal{B}^{s,0}},$$

for some  $c_0$  sufficiently small, (2.31) has a unique solution  $Y$  so that there holds

$$(2.34) \quad \begin{aligned} &\|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}^{s,0})} + \|\partial_3 Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{s,0})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+2,0})} + \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}^{s+1,0})} \\ &+ \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}^{s+1,0})} + \|Y_t\|_{L_t^1(\mathcal{B}^{s+2,0})} + \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{s,0})} \leq C(c_0 + \mathbf{g}(s)) \quad \text{for } s = 0, 3. \end{aligned}$$

If  $(Y_0, Y_1)$  satisfies moreover that

$$(2.35) \quad \begin{aligned} &\mathbf{g}(3) + \|\partial_3 Y_0\|_{\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}}} + \|Y_0\|_{\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}}} + \|Y_1\|_{\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}}} \\ &+ \|\partial_3 Y_0\|_{\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}}} + \|Y_0\|_{\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}}} + \|Y_1\|_{\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}}} \leq \delta_0 \end{aligned}$$

for some  $\delta_0$  sufficiently small, then the solution  $Y$  of (2.31) satisfies the following decay estimate

$$(2.36) \quad \begin{aligned} &\|Y_t(t)\|_{H^2} + \|\partial_3 Y(t)\|_{H^2} + \|\Delta Y(t)\|_{H^1} \\ &+ \langle t \rangle^{\frac{1}{8}} (\|\partial_3 Y_t(t)\|_{H^1} + \|\partial_3^2 Y(t)\|_{H^1} + \|\partial_3 Y(t)\|_{\dot{H}^2}) \leq C \langle t \rangle^{-\frac{1}{4}}. \end{aligned}$$

Let us remark that with more regularities on the initial data, we can study the decay rate of the solution in higher Sobolev norms. For a concise presentation, we shall not pursue this direction here.

### 3. ESTIMATES RELATED TO LITTLEWOOD-PALEY THEORY

The linearized system of (2.24) reads

$$(3.1) \quad \begin{cases} Y_{tt} - \Delta Y_t - \partial_3^2 Y = f, \\ Y|_{t=0} = Y_0, \quad Y_t|_{t=0} = Y_1. \end{cases}$$

As observed in [22, 21, 29], the corresponding symbolic equation to (3.1),

$$\lambda^2 + |\xi|^2 \lambda + \xi_3^2 = 0 \quad \text{for } \xi = (\xi_h, \xi_3) \quad \text{and} \quad \xi_h = (\xi_1, \xi_2),$$

has two different eigenvalues

$$(3.2) \quad \lambda_{\pm} = -\frac{|\xi|^2 \pm \sqrt{|\xi|^4 - 4\xi_3^2}}{2}.$$

The Fourier modes correspond to  $\lambda_+$  decays like  $e^{-t|\xi|^2}$ . Whereas the decay property of the Fourier modes corresponding to  $\lambda_-$  varies with directions of  $\xi$  as

$$(3.3) \quad \lambda_-(\xi) = -\frac{2\xi_3^2}{|\xi|^2(1 + \sqrt{1 - \frac{4\xi_3^2}{|\xi|^4}})} \rightarrow -1 \quad \text{as } |\xi| \rightarrow \infty$$

only in the  $\xi_3$  direction. Thus in order to capture this delicate decay property for the linear equation (3.1), we shall decompose our frequency space into two parts:  $\{\xi = (\xi_h, \xi_3) : |\xi|^2 \leq 2|\xi_3|\}$  and  $\{\xi = (\xi_h, \xi_3) : |\xi|^2 > 2|\xi_3|\}$ . This suggests to use anisotropic Littlewood-Paley theory in the analysis of (2.24).

In order to obtain a better description of the regularizing effect for the transport-diffusion equation, we will use Chemin-Lerner type spaces  $\tilde{L}_T^q(\dot{B}_{p,r}^s(\mathbb{R}^3))$  (see [2] for instance).

**Definition 3.1.** Let  $(r, q, p) \in [1, +\infty]^3$  and  $T \in (0, +\infty]$ . We define the norms of  $\tilde{L}_T^q(\dot{B}_{p,r}^s(\mathbb{R}^3))$  and  $\tilde{L}_T^q(\mathcal{B}^{s_1, s_2}(\mathbb{R}^3))$  by

$$\|u\|_{\tilde{L}_T^q(\dot{B}_{p,r}^s)} \stackrel{\text{def}}{=} \left( \sum_{j \in \mathbb{Z}} 2^{jrs} \|\Delta_j u\|_{L_T^q(L^p)}^r \right)^{\frac{1}{r}}, \quad \|u\|_{\tilde{L}_T^q(\mathcal{B}^{s_1, s_2})} \stackrel{\text{def}}{=} \sum_{j, \ell \in \mathbb{Z}^2} 2^{js_1} 2^{\ell s_2} \|\Delta_j \Delta_\ell^\vee u\|_{L_T^q(L^2)},$$

with the usual change if  $r = \infty$ .

The connection between the Besov space  $\dot{B}^s$  and the anisotropic Besov space  $\mathcal{B}^{s_1, s_2}$  can be illustrated by the following Lemma:

**Lemma 3.1** (Lemma 3.2 in [21] and [29]). Let  $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$ , which satisfy  $s_1 < \tau_1 + \tau_2 < s_2$  and  $\tau_2 > 0$ . Then  $a \in \mathcal{B}^{\tau_1, \tau_2}(\mathbb{R}^3)$  (resp.  $\tilde{L}_T^2(\mathcal{B}^{\tau_1, \tau_2})$ ) if  $a \in \dot{B}^{\tau_1 + \tau_2}$  (resp.  $a \in \tilde{L}_T^2(B^{\tau_1 + \tau_2})$ ) and there holds

$$(3.4) \quad \|a\|_{\mathcal{B}^{\tau_1, \tau_2}} \lesssim \|a\|_{B^{\tau_1 + \tau_2}} \quad \text{and} \quad \|u\|_{\tilde{L}_T^2(\mathcal{B}^{\tau_1, \tau_2})} \lesssim \|u\|_{\tilde{L}_T^2(B^{\tau_1 + \tau_2})}.$$

For the convenience of the readers, we recall the following Bernstein type lemma from [2, 11, 25]:

**Lemma 3.2.** Let  $\mathfrak{B}_h$  (resp.  $\mathfrak{B}_v$ ) be a ball of  $\mathbb{R}^2$  (resp.  $\mathbb{R}$ ), and  $\mathcal{C}_h$  (resp.  $\mathcal{C}_v$ ) a ring of  $\mathbb{R}^2$  (resp.  $\mathbb{R}$ ); let  $1 \leq p_2 \leq p_1 \leq \infty$  and  $1 \leq q_2 \leq q_1 \leq \infty$ . Then there holds:

If the support of  $\hat{a}$  is included in  $2^k \mathfrak{B}_h$ , then

$$\|\partial_h^\alpha a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{k(|\alpha| + 2(\frac{1}{p_2} - \frac{1}{p_1}))} \|a\|_{L_h^{p_2}(L_v^{q_1})}.$$

If the support of  $\hat{a}$  is included in  $2^\ell \mathfrak{B}_v$ , then

$$\|\partial_3^\beta a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{\ell(\beta + (\frac{1}{q_2} - \frac{1}{q_1}))} \|a\|_{L_h^{p_1}(L_v^{q_2})}.$$

If the support of  $\hat{a}$  is included in  $2^k \mathcal{C}_h$ , then

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-kN} \max_{|\alpha|=N} \|\partial_h^\alpha a\|_{L_h^{p_1}(L_v^{q_1})}.$$

If the support of  $\hat{a}$  is included in  $2^\ell \mathcal{C}_v$ , then

$$\|a\|_{L_h^{p_1}(L_v^{q_1})} \lesssim 2^{-\ell N} \|\partial_3^N a\|_{L_h^{p_1}(L_v^{q_1})}.$$

As applications of the above basic facts on Littlewood-Paley theory, we present the following product laws:

**Lemma 3.3** (Lemma 3.3 of [29]). Let  $s_1, s_2, \tau_1, \tau_2 \in \mathbb{R}$ , which satisfy  $s_1, s_2 \leq 1$ ,  $\tau_1, \tau_2 \leq \frac{1}{2}$  and  $s_1 + s_2 > 0$ ,  $\tau_1 + \tau_2 > 0$ . Then for  $a \in \mathcal{B}^{s_1, \tau_1}(\mathbb{R}^3)$  and  $b \in \mathcal{B}^{s_2, \tau_2}(\mathbb{R}^3)$ ,  $ab \in \mathcal{B}^{s_1 + s_2 - 1, \tau_1 + \tau_2 - \frac{1}{2}}(\mathbb{R}^3)$  and there holds

$$(3.5) \quad \|ab\|_{\mathcal{B}^{s_1 + s_2 - 1, \tau_1 + \tau_2 - \frac{1}{2}}} \lesssim \|a\|_{\mathcal{B}^{s_1, \tau_1}} \|b\|_{\mathcal{B}^{s_2, \tau_2}}.$$

**Remark 3.1.** Exactly along the same line to the proof of (3.5), we can show the following law of product that for any  $s > -1$

$$(3.6) \quad \begin{aligned} \|a \nabla b\|_{\mathcal{B}^{s, 0}} &\lesssim \|a\|_{\mathcal{B}^{1, \frac{1}{2}}} \|b\|_{\mathcal{B}^{s+1, 0}} + \|b\|_{\mathcal{B}^{1, \frac{1}{2}}} \|a\|_{\mathcal{B}^{s+1, 0}}, \\ \|ab\|_{\mathcal{B}^{s, 0}} &\lesssim \|a\|_{\mathcal{B}^{1, \frac{1}{2}}} \|b\|_{\mathcal{B}^{s, 0}} + \|b\|_{\mathcal{B}^{1, \frac{1}{2}}} \|a\|_{\mathcal{B}^{s, 0}}. \end{aligned}$$

We skip the details here.

**Lemma 3.4.** *Let  $s > -1$  and  $\delta \in [0, 1]$ , then one has*

$$(3.7) \quad \|ab\|_{\mathcal{B}_{1,\infty}^{s,-\frac{1}{2}}} \lesssim \|a\|_{\mathcal{B}^{1,0}} \|b\|_{\mathcal{B}^{s,0}} + \|a\|_{\mathcal{B}^{s+\delta,0}} \|b\|_{\mathcal{B}^{1-\delta,0}},$$

$$(3.8) \quad \|ab\|_{\mathcal{B}_{\infty,\infty}^{s',-\frac{1}{2}}} \lesssim \|a\|_{\mathcal{B}^{1,\frac{1}{2}}} \|b\|_{\mathcal{B}_{\infty,\infty}^{s',-\frac{1}{2}}} \quad \text{for } \forall s' \in [-1, 1].$$

*Proof.* Let us first recall the isentropic para-differential decomposition of Bony from [3]: let  $a, b \in \mathcal{S}'(\mathbb{R}^3)$ ,

$$(3.9) \quad \begin{aligned} ab &= T(a, b) + \bar{T}(a, b) + R(a, b), \quad \text{where} \\ T(a, b) &\stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} S_{j-1} a \Delta_j b, \quad \bar{T}(a, b) \stackrel{\text{def}}{=} T(b, a), \quad \text{and} \\ R(a, b) &\stackrel{\text{def}}{=} \sum_{j \in \mathbb{Z}} \Delta_j a \tilde{\Delta}_j b, \quad \text{with } \tilde{\Delta}_j b \stackrel{\text{def}}{=} \sum_{\ell=j-1}^{j+1} \Delta_\ell b. \end{aligned}$$

By using Bony's decomposition (3.9) for the whole space variables and the vertical variable simultaneously, we obtain

$$(3.10) \quad \begin{aligned} ab &= (T + \bar{T} + R)(T^v + \bar{T}^v + R^v)(a, b) \\ &= (TT^v + T\bar{T}^v + TR^v + \bar{T}T^v + \bar{T}\bar{T}^v + \bar{T}R^v + RT^v + R\bar{T}^v + RR^v)(a, b). \end{aligned}$$

In what follows, we shall deal with the typical terms above. We first deduce from Lemma 3.2 that

$$\begin{aligned} \|\Delta_j \Delta_\ell^v (TR^v(a, b))\|_{L^2} &\lesssim 2^{\frac{\ell}{2}} \sum_{\substack{|j'-j| \leq 4 \\ \ell' \geq \ell - N_0}} \|S_{j'-1} \Delta_{\ell'}^v a\|_{L_h^\infty(L_v^2)} \|\Delta_{j'} \Delta_{\ell'}^v b\|_{L^2} \\ &\lesssim 2^{\frac{\ell}{2}} \sum_{\substack{|j'-j| \leq 4 \\ \ell' \geq \ell - N_0}} d_{j', \ell'} 2^{-j's} \|a\|_{\mathcal{B}^{1,0}} \|b\|_{\mathcal{B}^{s,0}} \\ &\lesssim d_j 2^{-js} 2^{\frac{\ell}{2}} \|a\|_{\mathcal{B}^{1,0}} \|b\|_{\mathcal{B}^{s,0}} \end{aligned}$$

The same estimate holds for  $TT^v(a, b)$  and  $T\bar{T}^v(a, b)$ .

Similarly, we get, by applying Lemma 3.2, that

$$\begin{aligned} \|\Delta_j \Delta_\ell^v (\bar{T}R^v(a, b))\|_{L^2} &\lesssim 2^{\frac{\ell}{2}} \sum_{\substack{|j'-j| \leq 4 \\ \ell' \geq \ell - N_0}} \|\Delta_{j'} \Delta_{\ell'}^v a\|_{L^2} \|S_{j'-1} \Delta_{\ell'}^v b\|_{L_h^\infty(L_v^2)} \\ &\lesssim d_j 2^{-js} 2^{\frac{\ell}{2}} \|a\|_{\mathcal{B}^{s+\delta,0}} \|b\|_{\mathcal{B}^{1-\delta,0}} \end{aligned}$$

The same estimate holds for  $\bar{T}T^v(a, b)$  and  $\bar{T}\bar{T}^v(a, b)$ .

Finally due to  $s > -1$ , we have

$$\begin{aligned} \|\Delta_j \Delta_\ell^v (R\bar{R}^v(a, b))\|_{L^2} &\lesssim 2^j 2^{\frac{\ell}{2}} \sum_{\substack{j' \geq j - N_0 \\ \ell' \geq \ell - N_0}} \|\Delta_{j'} \Delta_{\ell'}^v a\|_{L^2} \|\tilde{\Delta}_{j'} \tilde{\Delta}_{\ell'}^v b\|_{L^2} \\ &\lesssim 2^j 2^{\frac{\ell}{2}} \sum_{\substack{j' \geq j - N_0 \\ \ell' \geq \ell - N_0}} d_{j', \ell'} 2^{-j'(s+1)} \|a\|_{\mathcal{B}^{1,0}} \|b\|_{\mathcal{B}^{s,0}} \\ &\lesssim d_j 2^{-js} 2^{\frac{\ell}{2}} \|a\|_{\mathcal{B}^{1,0}} \|b\|_{\mathcal{B}^{s,0}}. \end{aligned}$$

The same estimate holds for  $RT^v(a, b)$  and  $R\bar{T}^v(a, b)$ .

Hence in view of (3.10), we achieve (3.7). Exactly along the same line, we can prove (3.8), the detail of which is omitted.  $\square$

In order to prove the large time decay estimates of the solutions to (2.31), we need the following interpolation inequalities:

**Lemma 3.5.** *Let  $k \in \mathbb{N}$  and  $f \in \mathcal{S}(\mathbb{R}^3)$ . Then one has*

$$\begin{aligned}
(1) \quad & \|f\|_{L^2} \lesssim \|f\|_{\dot{B}_{2,\infty}^{0,-\frac{1}{2}}}^{\frac{2}{3}} \|\partial_3 f\|_{L^2}^{\frac{1}{3}}, \\
(2) \quad & \|\partial_3 f\|_{L^2} \lesssim \|f\|_{\dot{B}_{\infty,\infty}^{1,-\frac{1}{2}}}^{\frac{2}{3}} \|\nabla \partial_3 f\|_{L^2}^{\frac{1}{3}}, \\
(3) \quad & \|\nabla^k f\|_{L^2} \lesssim \|f\|_{\dot{B}_{2,\infty}^{\frac{3k}{2},-\frac{1}{2}}}^{\frac{2}{3}} \|\partial_3 f\|_{L^2}^{\frac{1}{3}} \quad \text{and} \\
(4) \quad & \|\nabla^k f\|_{L^2} \lesssim \|f\|_{\dot{B}_{2,\infty}^{\frac{3k-1}{2},-\frac{1}{2}}}^{\frac{2}{3}} \|\nabla \partial_3 f\|_{L^2}^{\frac{1}{3}}.
\end{aligned}$$

*Proof.* Note that by virtue of Definition 1.2, for any fixed integer  $N$ , one has

$$\begin{aligned}
(3.11) \quad & \|f\|_{L^2}^2 \sim \sum_{(j,\ell) \in \mathbb{Z}^2} \|\Delta_j \Delta_\ell^\vee f\|_{L^2}^2 = \sum_{\substack{\ell \leq N \\ j \in \mathbb{Z}}} \|\Delta_j \Delta_\ell^\vee f\|_{L^2}^2 + \sum_{\substack{\ell > N \\ j \in \mathbb{Z}}} \|\Delta_j \Delta_\ell^\vee f\|_{L^2}^2 \\
& \lesssim \sum_{\substack{\ell \leq N \\ j \in \mathbb{Z}}} c_j^2 2^\ell \|f\|_{\dot{B}_{2,\infty}^{0,-\frac{1}{2}}}^2 + \sum_{\substack{\ell > N \\ j \in \mathbb{Z}}} c_j^2 2^{-2\ell} \|\partial_3 f\|_{L^2}^2 \\
& \lesssim 2^N \|f\|_{\dot{B}_{2,\infty}^{0,-\frac{1}{2}}}^2 + 2^{-2N} \|\partial_3 f\|_{L^2}^2.
\end{aligned}$$

Here and in all that follows, we always denote  $(c_j)_{j \in \mathbb{Z}}$  to be a generic element of  $\ell^2(\mathbb{Z})$  so that  $\sum_{j \in \mathbb{Z}} c_j^2 = 1$ .

Let us now choose the integer  $N$  in (3.11) so that

$$2^N \sim \left( \frac{\|\partial_3 f\|_{L^2}^2}{\|f\|_{\dot{B}_{2,\infty}^{0,-\frac{1}{2}}}^2} \right)^{\frac{1}{3}},$$

leads to (1) of Lemma 3.5.

To prove (2) of Lemma 3.5, we first deduce from Proposition 2.22 of [2] that

$$(3.12) \quad \|\partial_3 f\|_{L^2} \lesssim \|\partial_3 f\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}}^{\frac{2}{3}} \|\nabla \partial_3 f\|_{L^2}^{\frac{1}{3}}.$$

Yet by virtue of Definition 1.1, we have

$$\begin{aligned}
\|\partial_3 f\|_{\dot{B}_{2,\infty}^{-\frac{1}{2}}} &= \sup_j 2^{-\frac{j}{2}} \|\Delta_j \partial_3 f\|_{L^2} \lesssim \sup_j 2^{-\frac{j}{2}} \sum_{\ell \leq j+N_0} \|\Delta_j \Delta_\ell^\vee \partial_3 f\|_{L^2} \\
&\lesssim \sup_j 2^{-\frac{j}{2}} \sum_{\ell \leq j+N_0} 2^{\frac{3\ell}{2}} 2^{-\frac{\ell}{2}} \|\Delta_j \Delta_\ell^\vee f\|_{L^2} \\
&\lesssim \sup_j 2^j \sup_\ell 2^{-\frac{\ell}{2}} \|\Delta_j \Delta_\ell^\vee f\|_{L^2} = \|f\|_{\dot{B}_{\infty,\infty}^{1,-\frac{1}{2}}}.
\end{aligned}$$

Resuming the above estimate into (3.12) gives rise to the second inequality of Lemma 3.5.

Along the same line to (3.11), for any integer  $k$ , we write

$$\begin{aligned}
\|\nabla^k f\|_{L^2}^2 &\sim \sum_{\substack{\ell-kj \leq N \\ j \in \mathbb{Z}}} 2^{2kj} \|\Delta_j \Delta_\ell^\vee f\|_{L^2}^2 + \sum_{\substack{\ell-kj > N \\ j \in \mathbb{Z}}} 2^{2kj} \|\Delta_j \Delta_\ell^\vee f\|_{L^2}^2 \\
&\lesssim \sum_{\substack{\ell-kj \leq N \\ j \in \mathbb{Z}}} c_j^2 2^{(\ell-kj)} \|f\|_{B_{2,\infty}^{\frac{3k}{2}, -\frac{1}{2}}}^2 + \sum_{\substack{\ell-kj > N \\ j \in \mathbb{Z}}} c_j^2 2^{-2(\ell-kj)} \|\partial_3 f\|_{L^2}^2 \\
&\lesssim 2^N \|f\|_{B_{2,\infty}^{\frac{3k}{2}, -\frac{1}{2}}}^2 + 2^{-2N} \|\partial_3 f\|_{L^2}^2.
\end{aligned}$$

Taking  $N$  in the above inequality so that

$$2^N \sim \left( \frac{\|\partial_3 f\|_{L^2}^2}{\|f\|_{B_{2,\infty}^{\frac{3k}{2}, -\frac{1}{2}}}^2} \right)^{\frac{1}{3}},$$

leads to (3) of Lemma 3.5.

Finally a direct application of (3) with  $f$  (resp.  $k$ ) there being replaced by  $\nabla f$  (resp.  $k-1$ ) leads to the last inequality of Lemma 3.5. This completes the proof of the lemma.  $\square$

**Lemma 3.6.** *Let  $s \in \mathbb{R}$  and  $b \in \mathcal{S}(\mathbb{R}^3)$ , one has*

$$(3.13) \quad \|b\|_{\dot{W}^{s,4}} \lesssim \|b\|_{B_{2,\infty}^{1+2s, -\frac{1}{2}}}^{\frac{1}{2}} \|\partial_3 b\|_{L^2}^{\frac{1}{2}}.$$

*Proof.* Note that  $\dot{B}_{p,2}^0 \hookrightarrow L^p$  for  $p \in [2, \infty[$  (see Theorem 2.40 of [2]), we have

$$\begin{aligned}
\|b\|_{\dot{W}^{s,4}} &= \| |D|^s b \|_{L^4} \lesssim \| |D|^s b \|_{\dot{B}_{4,2}^0} \\
&= \left( \sum_{j \in \mathbb{Z}} 2^{2js} \|\Delta_j b\|_{L^4}^2 \right)^{\frac{1}{2}} = \left( \sum_{j \in \mathbb{Z}} 2^{2js} \left\| \left( \sum_{\ell \in \mathbb{Z}} |\Delta_j \Delta_\ell^\vee b|^2 \right)^{\frac{1}{2}} \right\|_{L^4}^2 \right)^{\frac{1}{2}} \\
&\lesssim \left( \sum_{(j,\ell) \in \mathbb{Z}^2} 2^{2js} \|\Delta_j \Delta_\ell^\vee b\|_{L^4}^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

For any integer  $N$ , we get, by applying Lemma 3.2, that

$$\begin{aligned}
\sum_{(j,\ell) \in \mathbb{Z}^2} 2^{2js} \|\Delta_j \Delta_\ell^\vee b\|_{L^4}^2 &\lesssim \sum_{(j,\ell) \in \mathbb{Z}^2} 2^{j(1+2s)} 2^{\frac{\ell}{2}} \|\Delta_j \Delta_\ell^\vee b\|_{L^2}^2 \\
&\lesssim \sum_{\substack{\ell - \frac{2}{3}(1+2s)j \leq N \\ j \in \mathbb{Z}}} c_j^2 2^{\frac{3}{2}(\ell - \frac{2}{3}(1+2s)j)} \|b\|_{B_{2,\infty}^{1+2s, -\frac{1}{2}}}^2 \\
&\quad + \sum_{\substack{\ell - \frac{2}{3}(1+2s)j > N \\ j \in \mathbb{Z}}} c_j^2 2^{-\frac{3}{2}(\ell - \frac{2}{3}(1+2s)j)} \|\partial_3 b\|_{L^2}^2 \\
&\lesssim 2^{\frac{3}{2}N} \|b\|_{B_{2,\infty}^{1+2s, -\frac{1}{2}}}^2 + 2^{-\frac{3}{2}N} \|\partial_3 b\|_{L^2}^2.
\end{aligned}$$

Choosing  $N$  in the above inequality so that

$$2^N \sim \left( \frac{\|\partial_3 b\|_{L^2}^2}{\|b\|_{B_{2,\infty}^{1+2s, -\frac{1}{2}}}^2} \right)^{\frac{1}{3}}$$

leads to (3.13). This finishes the proof of the lemma.  $\square$

4.  $L_T^1(\mathcal{B}^{2, \frac{1}{2}})$  ESTIMATE OF  $\bar{Y}_t$ 

Let  $\bar{Y}$  be a smooth enough solution of (2.24) on  $[0, T]$ . The goal of this section is to present the *a priori*  $L_T^1(\text{Lip})$  estimate of  $\bar{Y}_t$ . Instead of handling the  $L_T^1(\mathcal{B}^{\frac{5}{2}, 0})$  norm of  $\bar{Y}_t$  as that in [21, 29], here we shall deal with the  $L_T^1(\mathcal{B}^{2, \frac{1}{2}})$  norm of  $\bar{Y}_t$ . For simplicity, we shall denote  $\text{div}_z = \text{div}$ ,  $\nabla_z = \nabla$  and  $\Delta_z = \Delta$  for short in this section.

**Lemma 4.1.** *Let  $Y$  is a smooth enough solution of (3.1) on  $[0, T]$ . Then for  $t \leq T$ , one has*

$$(4.1) \quad \begin{aligned} & \|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\partial_3 Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\Delta Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}^{1, \frac{1}{2}})} + \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}^{1, \frac{1}{2}})} \\ & + \|Y_t\|_{L_t^1(\mathcal{B}^{2, \frac{1}{2}})} \lesssim \|Y_1\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\partial_3 Y_0\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\Delta Y_0\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|f\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned}$$

*Proof.* The proof of this lemma basically follows from Proposition 4.1 of [21, 29]. For completeness, we present the details here. By applying the operator  $\Delta_j \Delta_\ell^\vee$  to (3.1) and then taking the  $L^2$  inner product of the resulting equation with  $\Delta_j \Delta_\ell^\vee Y_t$ , we write

$$(4.2) \quad \frac{1}{2} \frac{d}{dt} (\|\Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 Y\|_{L^2}^2) + \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 = (\Delta_j \Delta_\ell^\vee f | \Delta_j \Delta_\ell^\vee Y_t)_{L^2}.$$

Along the same line, one has

$$(\Delta_j \Delta_\ell^\vee Y_{tt} | \Delta \Delta_j \Delta_\ell^\vee Y) - \frac{1}{2} \frac{d}{dt} \|\Delta \Delta_j \Delta_\ell^\vee Y\|_{L^2}^2 - \|\partial_3 \nabla \Delta_j \Delta_\ell^\vee Y\|_{L^2}^2 = (\Delta_j \Delta_\ell^\vee f | \Delta \Delta_j \Delta_\ell^\vee Y).$$

Notice that

$$(\Delta_j \Delta_\ell^\vee Y_{tt} | \Delta \Delta_j \Delta_\ell^\vee Y) = \frac{d}{dt} (\Delta_j \Delta_\ell^\vee Y_t | \Delta \Delta_j \Delta_\ell^\vee Y) + \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2,$$

so that there holds

$$(4.3) \quad \begin{aligned} & \frac{d}{dt} \left( \frac{1}{2} \|\Delta \Delta_j \Delta_\ell^\vee Y\|_{L^2}^2 - (\Delta_j \Delta_\ell^\vee Y_t | \Delta \Delta_j \Delta_\ell^\vee Y) \right) \\ & - \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + \|\partial_3 \nabla \Delta_j \Delta_\ell^\vee Y\|_{L^2}^2 = -(\Delta_j \Delta_\ell^\vee f | \Delta \Delta_j \Delta_\ell^\vee Y). \end{aligned}$$

By summing up (4.2) with  $\frac{1}{4}$  of (4.3), we obtain

$$(4.4) \quad \begin{aligned} & \frac{d}{dt} g_{j, \ell}^2(t) + \frac{3}{4} \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + \frac{1}{4} \|\partial_3 \nabla \Delta_j \Delta_\ell^\vee Y\|_{L^2}^2 \\ & = (\Delta_j \Delta_\ell^\vee f | \Delta_j \Delta_\ell^\vee Y_t - \frac{1}{4} \Delta \Delta_j \Delta_\ell^\vee Y), \end{aligned}$$

where

$$\begin{aligned} g_{j, \ell}^2(t) & \stackrel{\text{def}}{=} \frac{1}{2} \left( \|\Delta_j \Delta_\ell^\vee Y_t(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 Y(t)\|_{L^2}^2 + \frac{1}{4} \|\Delta_j \Delta_\ell^\vee \Delta Y(t)\|_{L^2}^2 \right) \\ & - \frac{1}{4} (\Delta_j \Delta_\ell^\vee Y_t(t) | \Delta_j \Delta_\ell^\vee \Delta Y(t)). \end{aligned}$$

It is easy to observe that

$$(4.5) \quad g_{j, \ell}^2(t) \sim \|\Delta_j \Delta_\ell^\vee Y_t(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 Y(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \Delta Y(t)\|_{L^2}^2.$$

Now according to the heuristic analysis presented at the beginning of Section 3, we split the frequency analysis into the following two cases:

- When  $j \leq \frac{\ell+1}{2}$

In this case, one has

$$g_{j, \ell}^2(t) \sim \|\Delta_j \Delta_\ell^\vee Y_t(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 Y(t)\|_{L^2}^2,$$

and Lemma 3.2 implies that

$$\frac{3}{4} \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + \frac{1}{4} \|\partial_3 \nabla \Delta_j \Delta_\ell^\vee Y\|_{L^2}^2 \geq c 2^{2j} (\|\Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 Y\|_{L^2}^2).$$

Hence it follows from (4.4) that

$$\begin{aligned}
 (4.6) \quad & \|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^\infty(L^2)} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y\|_{L_t^\infty(L^2)} + \|\Delta_j \Delta_\ell^\vee \Delta Y\|_{L_t^\infty(L^2)} \\
 & + 2^{2j} (\|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^1(L^2)} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y\|_{L_t^1(L^2)}) \\
 & \lesssim \|\Delta_j \Delta_\ell^\vee Y_1\|_{L^2} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y_0\|_{L^2} + \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}.
 \end{aligned}$$

• When  $j > \frac{\ell+1}{2}$

In this case, we have

$$g_{j,\ell}^2(t) \sim \|\Delta_j \Delta_\ell^\vee Y_t(t)\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \Delta Y(t)\|_{L^2}^2$$

and Lemma 3.2 implies that

$$\begin{aligned}
 \frac{3}{4} \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + \frac{1}{4} \|\partial_3 \nabla \Delta_j \Delta_\ell^\vee Y\|_{L^2}^2 & \geq c(2^{2j} \|\Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + 2^{2j} 2^{2\ell} \|\Delta_j \Delta_\ell^\vee Y\|_{L^2}^2) \\
 & \geq c \frac{2^{2\ell}}{2^{2j}} (\|\Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \Delta Y\|_{L^2}^2).
 \end{aligned}$$

Then we deduce from (4.4) that

$$\begin{aligned}
 (4.7) \quad & \|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^\infty(L^2)} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y\|_{L_t^\infty(L^2)} + \|\Delta_j \Delta_\ell^\vee \Delta Y\|_{L_t^\infty(L^2)} \\
 & + c \left( \frac{2^{2\ell}}{2^{2j}} \|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^1(L^2)} + 2^{2\ell} \|\Delta_j \Delta_\ell^\vee Y\|_{L_t^1(L^2)} \right) \\
 & \lesssim \|\Delta_j \Delta_\ell^\vee Y_1\|_{L^2} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y_0\|_{L^2} + \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}.
 \end{aligned}$$

On the other hand, it is easy to observe from (3.1) that

$$\frac{1}{2} \frac{d}{dt} \|\Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 + \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L^2}^2 = (\partial_3^2 \Delta_j \Delta_\ell^\vee Y + \Delta_j \Delta_\ell^\vee f \mid \Delta_j \Delta_\ell^\vee Y_t)_{L^2},$$

from which, Lemma 3.2 and (4.7), we deduce that for  $j > \frac{\ell+1}{2}$

$$\begin{aligned}
 (4.8) \quad & \|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^\infty(L^2)} + 2^{2j} \|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^1(L^2)} \\
 & \lesssim \|\Delta_j \Delta_\ell^\vee Y_1\|_{L^2} + 2^{2\ell} \|\Delta_j \Delta_\ell^\vee Y\|_{L_t^1(L^2)} + \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)} \\
 & \lesssim \|\Delta_j \Delta_\ell^\vee Y_1\|_{L^2} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y_0\|_{L^2} + \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}.
 \end{aligned}$$

In view of (4.6)-(4.8), we obtain for all  $(j, \ell) \in \mathbb{Z}^2$ , that

$$\begin{aligned}
 (4.9) \quad & \|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^\infty(L^2)} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y\|_{L_t^\infty(L^2)} + \|\Delta_j \Delta_\ell^\vee \Delta Y\|_{L_t^\infty(L^2)} + 2^{2j} \|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^1(L^2)} \\
 & \lesssim \|\Delta_j \Delta_\ell^\vee Y_1\|_{L^2} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y_0\|_{L^2} + \|\Delta_j \Delta_\ell^\vee \Delta Y_0\|_{L^2} + \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}.
 \end{aligned}$$

Whereas by integrating (4.4) over  $[0, t]$ , we get

$$\begin{aligned}
 & \|g_{j,\ell}^2\|_{L^\infty(0,t)} + \frac{3}{4} \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L_t^2(L^2)}^2 + \frac{1}{4} \|\nabla \Delta_j \Delta_\ell^\vee \partial_3 Y\|_{L_t^2(L^2)}^2 \\
 & \leq \|\Delta_j \Delta_\ell^\vee Y_1\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 Y_0\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \Delta Y_0\|_{L^2}^2 \\
 & \quad + \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)} (\|\Delta_j \Delta_\ell^\vee Y_t\|_{L_t^\infty(L^2)} + \frac{1}{4} \|\Delta_j \Delta_\ell^\vee Y\|_{L_t^\infty(L^2)}) \\
 & \leq \|\Delta_j \Delta_\ell^\vee Y_1\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \partial_3 Y_0\|_{L^2}^2 + \|\Delta_j \Delta_\ell^\vee \Delta Y_0\|_{L^2}^2 + C \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}^2 + \frac{1}{2} \|g_{j,\ell}^2\|_{L^\infty(0,t)},
 \end{aligned}$$

which in particular gives rise to

$$\begin{aligned}
 & \|\nabla \Delta_j \Delta_\ell^\vee Y_t\|_{L_t^2(L^2)} + \|\nabla \Delta_j \Delta_\ell^\vee \partial_3 Y\|_{L_t^2(L^2)} \\
 & \lesssim \|\Delta_j \Delta_\ell^\vee Y_1\|_{L^2} + \|\Delta_j \Delta_\ell^\vee \partial_3 Y_0\|_{L^2} + \|\Delta_j \Delta_\ell^\vee \Delta Y_0\|_{L^2} + \|\Delta_j \Delta_\ell^\vee f\|_{L_t^1(L^2)}.
 \end{aligned}$$

Summing up the above inequality with (4.9) and multiplying the inequality by  $2^{\frac{\ell}{2}}$  and then summing up the resulting inequality for  $(j, \ell) \in \mathbb{Z}^2$ , we achieve (4.1). This completes the proof of the lemma.  $\square$

**Proposition 4.1.** *Let  $\bar{Y}$  is a smooth enough solution of (2.24) on  $[0, T]$ . Then there exist sufficiently small positive constants,  $c_0, \varepsilon_0$ , so that if*

$$(4.10) \quad \|Y_1\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\partial_3 \bar{Y}_0\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\bar{Y}_0\|_{\mathcal{B}^{2, \frac{1}{2}}} \leq c_0 \quad \text{and} \quad \varepsilon \leq \varepsilon_0,$$

we have

$$(4.11) \quad \begin{aligned} & \|\bar{Y}_t\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\partial_3 \bar{Y}\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\bar{Y}\|_{\tilde{L}_t^\infty(\mathcal{B}^{2, \frac{1}{2}})} + \|\bar{Y}_t\|_{\tilde{L}_t^2(\mathcal{B}^{1, \frac{1}{2}})} + \|\partial_3 \bar{Y}\|_{\tilde{L}_t^2(\mathcal{B}^{1, \frac{1}{2}})} \\ & + \|\bar{Y}_t\|_{L_t^1(\mathcal{B}^{2, \frac{1}{2}})} + \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})} \leq C \left( \|\bar{Y}_1\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\partial_3 \bar{Y}_0\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\bar{Y}_0\|_{\mathcal{B}^{2, \frac{1}{2}}} \right) \end{aligned}$$

for any  $t \leq T$ .

*Proof.* Let us denote

$$(4.12) \quad T^* \stackrel{\text{def}}{=} \sup \{ t \in [0, T] : \|\nabla \bar{Y}\|_{L_t^\infty(\mathcal{B}^{1, \frac{1}{2}})} \leq \delta \}.$$

We shall prove that for  $\varepsilon_0$  and  $c_0$  sufficiently small,  $T^* = T$ .

According to Lemma 4.1, it remains to estimate  $\|f\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})}$  for  $f$  given by (2.25). Toward this and in view of (2.25), we decompose  $f$  as

$$(4.13) \quad \begin{aligned} f &= f_1 + f_2 + f_3 \quad \text{with} \\ f_1 &\stackrel{\text{def}}{=} \mathcal{B}^t \nabla \cdot ((\mathcal{A} - Id)(\mathcal{A} - Id)^t + (\mathcal{A} - Id) + (\mathcal{A} - Id)^t) \mathcal{B}^t \nabla \bar{Y}_t, \\ f_2 &\stackrel{\text{def}}{=} \mathcal{B}^t \nabla \cdot ((\mathcal{B} - Id)^t \nabla \bar{Y}_t) + (\mathcal{B} - Id)^t \Delta \bar{Y}_t, \\ f_3 &\stackrel{\text{def}}{=} -(\mathcal{B}\mathcal{A})^t \nabla \mathbf{p}, \end{aligned}$$

where the matrix  $\mathcal{B}$  and  $\nabla \mathbf{p}$  are determined respectively by (2.21) and (2.30).

On the other hand, in view of (2.15) and (2.16), for  $b_0 = e_3 + \varepsilon \phi$  with  $\phi$  satisfying (2.14), we have

$$\begin{aligned} |z_3 - w_3(z)| &\leq \varepsilon \int_0^{w_3(z)} \frac{|\phi_3(y_h(w_h, w'_3), w'_3)|}{1 - \varepsilon |\phi_3(y_h(w_h, w'_3), w'_3)|} dw'_3 \\ &\leq 2\varepsilon \int_0^K |\phi_3(y_h(w_h, w'_3), w'_3)| dw'_3 \leq 2\varepsilon K \|\phi_3\|_{L^\infty}, \end{aligned}$$

whenever  $\varepsilon \leq \frac{1}{2\|\phi_3\|_{L^\infty}}$ . This proves that as long as  $\varepsilon \leq \varepsilon_1 \stackrel{\text{def}}{=} \min \left( \frac{1}{4\|\phi_3\|_{L^\infty}}, \frac{1}{4K\|\phi_3\|_{L^\infty}} \right)$ , there holds

$$(4.14) \quad |z_3 - w_3(z)| \leq \frac{1}{2}.$$

Now for  $K$  given by (2.14), let us introduce a smooth cut-off function  $\eta(z_3)$  so that  $\eta(z_3) = \begin{cases} 0, & z_3 \geq 2 + K, \\ 1, & -1 \leq z_3 \leq 1 + K, \\ 0, & z_3 \leq -2. \end{cases}$  Then thanks to (2.15), (2.16) and (4.14), we split  $A_2(y(w(z)))$  given by (2.18) as

$$(4.15) \quad A_2(y(w(z))) = A_{2,1}(z) + A_{2,2}(z) \quad \text{with} \quad A_{2,2}(z) \stackrel{\text{def}}{=} (1 - \eta(z_3))A_2^h(z_h),$$



and

$$A_{2,1}(z) \stackrel{\text{def}}{=} \eta(z_3) \begin{pmatrix} \int_0^{w_3(z)} \frac{\partial}{\partial y_1} \left( \frac{b_0^1}{b_0^3} \right) (y_h(z_h, y_3), y_3) dy_3 & \int_0^{w_3(z)} \frac{\partial}{\partial y_2} \left( \frac{b_0^1}{b_0^3} \right) (y_h(z_h, y_3), y_3) dy_3 & 0 \\ \int_0^{w_3(z)} \frac{\partial}{\partial y_1} \left( \frac{b_0^2}{b_0^3} \right) (y_h(z_h, y_3), y_3) dy_3 & \int_0^{w_3(z)} \frac{\partial}{\partial y_2} \left( \frac{b_0^2}{b_0^3} \right) (y_h(z_h, y_3), y_3) dy_3 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$A_2^h(z_h) \stackrel{\text{def}}{=} \begin{pmatrix} \int_0^K \frac{\partial}{\partial y_1} \left( \frac{b_0^1}{b_0^3} \right) (y_h(z_h, y_3), y_3) dy_3 & \int_0^K \frac{\partial}{\partial y_2} \left( \frac{b_0^1}{b_0^3} \right) (y_h(z_h, y_3), y_3) dy_3 & 0 \\ \int_0^K \frac{\partial}{\partial y_1} \left( \frac{b_0^2}{b_0^3} \right) (y_h(z_h, y_3), y_3) dy_3 & \int_0^K \frac{\partial}{\partial y_2} \left( \frac{b_0^2}{b_0^3} \right) (y_h(z_h, y_3), y_3) dy_3 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Similarly, we decompose  $A_3(w(z))$ , which is given by (2.20), as

$$(4.16) \quad A_3(w(z)) = A_{3,1}(z) + A_{3,2}(z) \quad \text{with} \quad A_{3,2}(z) \stackrel{\text{def}}{=} (1 - \eta(z_3)) A_3^h(z_h),$$

and

$$A_{3,1}(z) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \eta(z_3) \int_0^{w_3(z)} \frac{\partial}{\partial z_1} \left( \frac{1}{b_0^3(y_h(z_h, y_3), y_3)} \right) dy_3 & \eta(z_3) \int_0^{w_3(z)} \frac{\partial}{\partial z_2} \left( \frac{1}{b_0^3(y_h(z_h, y_3), y_3)} \right) dy_3 & \frac{1}{b_0^3} \end{pmatrix},$$

$$A_3^h(z_h) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \int_0^K \frac{\partial}{\partial z_1} \left( \frac{1}{b_0^3(y_h(z_h, y_3), y_3)} \right) dy_3 & \int_0^K \frac{\partial}{\partial z_2} \left( \frac{1}{b_0^3(y_h(z_h, y_3), y_3)} \right) dy_3 & 0 \end{pmatrix}.$$

Then by virtue of (2.21), (4.15) and (4.16), we find

$$(4.17) \quad \mathcal{B} - Id = \underbrace{(\mathfrak{A}_1^{-1} - Id) + (A_{3,1} - Id) + (\mathfrak{A}_1^{-1} - Id)(A_{3,1} - Id) + A_{3,1} \mathfrak{A}_1^{-1} A_{2,1}}_{\mathcal{B}_1} \\ + \underbrace{A_{3,2} \mathfrak{A}_1^{-1} + A_{3,1} \mathfrak{A}_1^{-1} A_{2,2} + A_{3,2} \mathfrak{A}_1^{-1} (A_{2,1} + A_{2,2})}_{\mathcal{B}_2},$$

where  $\mathfrak{A}_1(z) \stackrel{\text{def}}{=} A_1(y_h(z_h, w_3(z)), w_3(z))$ .

**Lemma 4.2.** *Under the assumptions of Theorem 1.1, there exists a sufficiently small constant  $\varepsilon_2 \leq \varepsilon_1$ , which depends on  $\|\nabla \phi\|_{W^{2,\infty}}$ ,  $\|\nabla \phi\|_{H^2}$ , and  $\|\nabla_h \phi\|_{L_v^\infty(H_h^2)}$ , so that for  $\varepsilon \leq \varepsilon_2$ , we have*

$$(4.18) \quad \|\mathfrak{A}_1 - Id\|_{\mathcal{B}^{1,\frac{1}{2}}} + \|A_{2,1}\|_{\mathcal{B}^{1,\frac{1}{2}}} + \|A_{3,1} - Id\|_{\mathcal{B}^{1,\frac{1}{2}}} + \|A_2^h\|_{\dot{B}_h^1} + \|A_3^h\|_{\dot{B}_h^1} \\ \leq C\varepsilon (\|\nabla \phi\|_{H^2} + \|\nabla_h \phi\|_{L_v^\infty(H_h^2)}) \leq 1.$$

While it follows from (4.14) and the definition of the cut-off function,  $\eta(z_3)$ , that

$$(1 - \eta(z_3)) \int_{-1}^{z_3} (e_3 - b_0(y_h(z_h, w_3(z_h, z'_3)), w_3(z_h, z'_3))) dz'_3 \\ = (1 - \eta(z_3)) \int_{-1}^{K+1} (e_3 - b_0(y_h(z_h, w_3(z_h, z'_3)), w_3(z_h, z'_3))) dz'_3,$$

so that we deduce from (2.21) and (2.23) that

$$(4.19) \quad \tilde{Y}(z) = \int_{-1}^{z_3} (e_3 - b_0(y(w(z_h, z'_3)))) dz'_3 - \int_{-1}^{K+1} (e_3 - b_0(y(w(z_h, z'_3)))) dz'_3 \\ = \eta(z_3) \left( \int_{-1}^{z_3} (e_3 - b_0(y(w(z_h, z'_3)))) dz'_3 - \int_{-1}^{K+1} (e_3 - b_0(y(w(z_h, z'_3)))) dz'_3 \right).$$

**Lemma 4.3.** *Under the assumptions of Theorem 1.1 and for  $\varepsilon \leq \varepsilon_3$  with  $\varepsilon_3 \leq \varepsilon_2$  and depending on  $\|\nabla\phi\|_{W^{2,\infty}}$ ,  $\|\phi\|_{H^3}$  and  $\|\phi\|_{L^\infty(H_h^3)}$ , one has*

$$(4.20) \quad \|\partial_3 \tilde{Y}\|_{\mathcal{B}^{0,\frac{1}{2}}} + \|\nabla \tilde{Y}\|_{\mathcal{B}^{1,\frac{1}{2}}} \leq C\varepsilon(\|\phi\|_{H^3} + \|\phi\|_{L^\infty(H_h^3)}) \leq 1.$$

We shall postpone the proof of the above two lemmas in the Appendix A.

With the above preparations, we now present the estimate of  $\|f\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})}$ . Since the estimate to all the terms in  $f_1$  and  $f_2$  given by (4.13) are the same type, let us present the detailed estimate to the following term:

$$\mathcal{B}^t \nabla \cdot ((\mathcal{A} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t) = (Id + \mathcal{B}_1^t + \mathcal{B}_2^t) \nabla \cdot ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t$$

for  $\mathcal{B}_1, \mathcal{B}_2$  given by (4.17).

It follows from the law of product, Lemma 3.3, and (4.18) that for  $\varepsilon \leq \varepsilon_2$ ,

$$\begin{aligned} & \left\| (Id + \mathcal{B}_1^t) \nabla \cdot ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\ & \lesssim (1 + \|\mathcal{B}_1\|_{\mathcal{B}^{1,\frac{1}{2}}}) \left\| ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})} \\ & \lesssim \left\| ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})}, \end{aligned}$$

and

$$\begin{aligned} & \left\| (A_{3,2} \mathfrak{A}_1^{-1})^t \nabla \cdot ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\ & \lesssim (1 + \|\mathfrak{A}_1 - Id\|_{\mathcal{B}^{1,\frac{1}{2}}}) \|A_{3,2}^h\|_{\dot{B}_h^1} \left\| (1 - \eta(z_3)) \nabla \cdot ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\ & \lesssim \left\| \nabla \cdot ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})}. \end{aligned}$$

Along the same line, one can show that similar estimate holds with  $A_{3,2} \mathfrak{A}_1^{-1}$  in the above inequality being replaced by the other terms in  $\mathcal{B}_2$ . This proves that

$$(4.21) \quad \left\| \mathcal{B}^t \nabla \cdot ((\mathcal{A} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t) \right\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \lesssim \left\| ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})}.$$

Using the fact that  $(Id + A)^{-1} - Id = \sum_{n=1}^{\infty} A^n$  and (4.12), (4.20), we deduce by the law of product, Lemma 3.3, that for  $t \leq T^*$  and  $\varepsilon \leq \varepsilon_3$ ,

$$\begin{aligned} & \left\| ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})} \\ & \leq C(\|\nabla \bar{Y}\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})} + \|\nabla \tilde{Y}\|_{\mathcal{B}^{1,\frac{1}{2}}}) \left\| \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})} \\ & \leq C(\delta + \varepsilon)(1 + \|\phi\|_{H^3} + \|\phi\|_{L^\infty(H_h^3)}) \left\| \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})}. \end{aligned}$$

Whereas the proof of (4.21) ensures that

$$\left\| \mathcal{B}^t \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})} \lesssim \left\| \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})}.$$

Hence, by virtue of (4.21), we infer for  $t \leq T^*$  and  $\varepsilon \leq \varepsilon_3$

$$(4.22) \quad \left\| \mathcal{B}^t \nabla \cdot ((\mathcal{A} - Id)^t \mathcal{B}^t \nabla \bar{Y}_t) \right\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \leq C(\varepsilon + \delta)(1 + \|\phi\|_{H^3} + \|\phi\|_{L^\infty(H_h^3)}) \left\| \nabla \bar{Y}_t \right\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})}.$$

The same estimate holds for  $f_1$  and  $f_2$  given by (4.13).

In order to deal with the estimate of  $f_3$  given by (4.13), we need the following lemma concerning the estimate of the pressure function:

**Lemma 4.4.** *Let  $t \leq T^*$  and  $\varepsilon \leq \bar{\varepsilon} \leq \varepsilon_3, \delta \leq \bar{\delta}$  for some sufficiently small constants  $\bar{\varepsilon}$  and  $\bar{\delta}$ . Then there holds*

$$(4.23) \quad \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})} \leq C \left( \|\partial_3 \bar{Y}\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})}^2 + \|\bar{Y}_t\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})}^2 \right).$$

Let us postpone the proof of this lemma after the proof of the proposition.

In view of (4.13), we get, by a similar proof of (4.21), that

$$\begin{aligned} \|f_3\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})} &\lesssim \|(Id + ((Id + \nabla \bar{Y} + \nabla \tilde{Y})^{-1} - Id)) \nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})} \\ &\lesssim (1 + \|\nabla \bar{Y}\|_{L_t^\infty(\mathcal{B}^{1, \frac{1}{2}})} + \|\nabla \tilde{Y}\|_{\mathcal{B}^{1, \frac{1}{2}}}) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})}. \end{aligned}$$

Hence by virtue of (4.12), (4.20) and Lemma 4.4, we obtain for  $t \leq T^*$  and  $\varepsilon \leq \bar{\varepsilon}, \delta \leq \bar{\delta}$ ,

$$\|f_3\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})} \leq C \left( \|\partial_3 \bar{Y}\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})}^2 + \|\bar{Y}_t\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})}^2 \right),$$

from which, (4.13) and (4.22), we deduce that for  $t \leq T^*$  and  $\varepsilon \leq \bar{\varepsilon}, \delta \leq \bar{\delta}$ ,

$$(4.24) \quad \begin{aligned} \|f\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})} &\leq C \left( (\varepsilon + \delta) (1 + \|\phi\|_{H^3} + \|\phi\|_{L_v^\infty(H_h^3)}) \|\bar{Y}_t\|_{L_t^1(\mathcal{B}^{2, \frac{1}{2}})} \right. \\ &\quad \left. + \|\bar{Y}_t\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})}^2 + \|\partial_3 \bar{Y}\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})}^2 \right). \end{aligned}$$

Let us denote

$$(4.25) \quad \begin{aligned} \varepsilon_0 &\stackrel{\text{def}}{=} \min(\bar{\varepsilon}, \frac{1}{4C} (1 + \|\phi\|_{H^3} + \|\phi\|_{L_v^\infty(H_h^3)})^{-1}) \quad \text{and} \\ \delta_0 &\stackrel{\text{def}}{=} \min(\bar{\delta}, \frac{1}{4C} (1 + \|\phi\|_{H^3} + \|\phi\|_{L_v^\infty(H_h^3)})^{-1}). \end{aligned}$$

Then we deduce from (4.1), (4.23) and (4.24) that for  $t \leq T^*$  and  $\varepsilon \leq \varepsilon_0, \delta \leq \delta_0$ ,

$$(4.26) \quad \begin{aligned} &\|\bar{Y}_t\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\partial_3 \bar{Y}\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\Delta \bar{Y}\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\bar{Y}_t\|_{\tilde{L}_t^2(\mathcal{B}^{1, \frac{1}{2}})} \\ &\quad + \|\partial_3 \bar{Y}\|_{\tilde{L}_t^2(\mathcal{B}^{1, \frac{1}{2}})} + \|\bar{Y}_t\|_{L_t^1(\mathcal{B}^{2, \frac{1}{2}})} + \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})} \\ &\leq C \left( \|Y_1\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\partial_3 \bar{Y}_0\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\Delta \bar{Y}_0\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\bar{Y}_t\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})}^2 + \|\partial_3 \bar{Y}\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})}^2 \right). \end{aligned}$$

Let us denote

$$T^* \stackrel{\text{def}}{=} \sup \{ t \leq T^*, \quad \|\bar{Y}_t\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})} + \|\partial_3 \bar{Y}\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})} \leq 2Cc_0 \}.$$

Then we deduce from (4.10) and (4.26) that for  $t \leq T^*$ ,

$$\|\bar{Y}_t\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})} + \|\partial_3 \bar{Y}\|_{L_t^2(\mathcal{B}^{1, \frac{1}{2}})} \leq \frac{Cc_0}{1 - 2C^2c_0} \leq \frac{3}{2}Cc_0,$$

provided that  $c_0 \leq \frac{1}{6C^2}$ . This proves that  $T^* = T^*$ , and there holds

$$\begin{aligned} &\|\bar{Y}_t\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\partial_3 \bar{Y}\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\Delta \bar{Y}\|_{\tilde{L}_t^\infty(\mathcal{B}^{0, \frac{1}{2}})} + \|\bar{Y}_t\|_{\tilde{L}_t^2(\mathcal{B}^{1, \frac{1}{2}})} \\ &\quad + \|\partial_3 \bar{Y}\|_{\tilde{L}_t^2(\mathcal{B}^{1, \frac{1}{2}})} + \|\bar{Y}_t\|_{L_t^1(\mathcal{B}^{2, \frac{1}{2}})} + \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0, \frac{1}{2}})} \\ &\leq \frac{C}{1 - 2Cc_0} \left( \|Y_1\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\partial_3 \bar{Y}_0\|_{\mathcal{B}^{0, \frac{1}{2}}} + \|\Delta \bar{Y}_0\|_{\mathcal{B}^{0, \frac{1}{2}}} \right) \end{aligned}$$

for any  $t \leq T^*$ . Then for  $c_0 \leq \min(\frac{1}{6C^2}, \frac{1}{8C})$ , by taking  $\delta = \min(\delta_0, 2Cc_0)$ , for  $\delta_0$  given by (4.25), in (4.12) shows that  $T^* = T$  and (4.11) holds for any  $t \leq T$ . This completes the proof of Proposition 4.1.  $\square$

Let us now present the proof of Lemma 4.4.

*Proof of Lemma 4.4.* We first deduce from (2.30) that

$$\begin{aligned}
 \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} &\lesssim \|\det(\mathcal{B}^{-1})(\mathcal{B}\mathcal{A}\mathcal{A}^t\mathcal{B}^t - Id)\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\
 (4.27) \quad &+ \|(\det(\mathcal{B}^{-1})Id - Id)\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\
 &+ \|\mathcal{B}\mathcal{A}\text{div}(\det(\mathcal{B}^{-1})\mathcal{B}\mathcal{A}(\partial_3\bar{Y} \otimes \partial_3\bar{Y} - \bar{Y}_t \otimes \bar{Y}_t))\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})}.
 \end{aligned}$$

Note that

$$\begin{aligned}
 \mathcal{B}\mathcal{A}\mathcal{A}^t\mathcal{B}^t - Id &= (\mathcal{A}\mathcal{A}^t - Id) + (\mathcal{B} - Id) + (\mathcal{B} - Id)^t \\
 (4.28) \quad &+ (\mathcal{B} - Id)(\mathcal{A}\mathcal{A}^t - Id) + (\mathcal{B} - Id)(\mathcal{B} - Id)^t \\
 &+ (\mathcal{A}\mathcal{A}^t - Id)(\mathcal{B} - Id)^t + (\mathcal{B} - Id)(\mathcal{A}\mathcal{A}^t - Id)(\mathcal{B} - Id)^t.
 \end{aligned}$$

Let us deal with the typical term above. Indeed it follows Lemma 3.3 that

$$\begin{aligned}
 \|(\mathcal{A}\mathcal{A}^t - Id)(\mathcal{B} - Id)^t\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} &\lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})} \|(\mathcal{B} - Id)^t\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\
 (4.29) \quad &\lesssim (\|\nabla \bar{Y}\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})} + \|\nabla \tilde{Y}\|_{\mathcal{B}^{1,\frac{1}{2}}}) \|(\mathcal{B} - Id)^t\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})}.
 \end{aligned}$$

And for  $\mathcal{B}_1$  given by (4.17), one has

$$\|\mathcal{B}_1^t\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \lesssim \|\mathcal{B}_1\|_{\mathcal{B}^{1,\frac{1}{2}}} \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})},$$

and Lemmas 3.3 and 4.2 ensure that for  $\varepsilon \leq \varepsilon_2$

$$\begin{aligned}
 \|(A_{3,2}\mathfrak{A}_1^{-1})^t\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} &\lesssim (1 + \|\mathfrak{A}_1 - Id\|_{\mathcal{B}^{1,\frac{1}{2}}}) \|A_3^h\|_{\dot{B}_h^1} \|(1 - \eta(z_3))\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\
 &\lesssim \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})}.
 \end{aligned}$$

The same estimate holds with  $A_{3,2}\mathfrak{A}_1^{-1}$  in the above inequality being replaced by the other terms in  $\mathcal{B}_2$  given by (4.17). This leads to

$$\|\mathcal{B}_2^t\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \lesssim \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})},$$

which together with (4.12), (4.20) and (4.29) ensures that for  $\varepsilon \leq \varepsilon_3$

$$\|(\mathcal{B} - Id)^t\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \lesssim \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})},$$

and

$$(4.30) \quad \|(\mathcal{A}\mathcal{A}^t - Id)(\mathcal{B} - Id)^t\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \leq C(\varepsilon + \delta)(1 + \|\phi\|_{H^3} + \|\phi\|_{L_v^\infty(H_h^3)}) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})}.$$

Similar estimate holds for the other terms in (4.28). Furthermore, due to the special structure of the matrix  $\mathcal{B}$  given by (4.17), we get, by a similar derivation of (4.30), that

$$(4.31) \quad \|\det(\mathcal{B}^{-1})(\mathcal{B}\mathcal{A}\mathcal{A}^t\mathcal{B}^t - Id)\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \leq C(\varepsilon + \delta)(1 + \|\phi\|_{H^3} + \|\phi\|_{L_v^\infty(H_h^3)}) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})}.$$

The same estimate holds for  $\|(\det(\mathcal{B}^{-1})Id - Id)\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})}$ .

Along the same line, we can show that for  $\varepsilon \leq \varepsilon_3$ ,

$$\|\mathcal{B}\mathcal{A}\text{div}(\det(\mathcal{B}^{-1})\mathcal{B}\mathcal{A}(\partial_3\bar{Y} \otimes \partial_3\bar{Y} - \bar{Y}_t \otimes \bar{Y}_t))\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \leq C\left(\|\partial_3\bar{Y}\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}^2 + \|\bar{Y}_t\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}^2\right),$$

from which and (4.31), we infer

$$\|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \leq C\left((\varepsilon + \delta)(1 + \|\phi\|_{H^3} + \|\phi\|_{L_v^\infty(H_h^3)}) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} + \|\partial_3\bar{Y}\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}^2 + \|\bar{Y}_t\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}^2\right),$$

which leads to (4.23) by taking  $\varepsilon \leq \bar{\varepsilon}$  and  $\delta \leq \bar{\delta}$  with  $\bar{\varepsilon}$  and  $\bar{\delta}$  being given by

$$\bar{\varepsilon} \stackrel{\text{def}}{=} \min(\varepsilon_3, \bar{\delta}) \quad \text{and} \quad \bar{\delta} \stackrel{\text{def}}{=} \frac{1}{4C}(1 + \|\phi\|_{H^3} + \|\phi\|_{L_v^\infty(H_h^3)})^{-1}.$$

This completes the proof of Lemma 4.4.  $\square$

### 5. THE DECAY OF THE SOLUTIONS TO (2.31)

In this section, let us fix  $b_0 = e_3$ , then the matrix  $\mathcal{B}$  given by (2.21) equals to  $Id$ . Then the System (2.12) then becomes (2.31). For simplicity, we shall denote  $\nabla_y$  by  $\nabla$  in this section.

**Proposition 5.1.** *Let  $Y$  be a smooth global solution of (2.31). Let*

$$E_0 \stackrel{\text{def}}{=} \|Y_1\|_{H^1}^2 + \|\partial_3 Y_0\|_{H^1}^2 + \|\Delta Y_0\|_{L^2}^2.$$

*If we assume that*

$$(5.1) \quad E_0 + \sup_{t \in \mathbb{R}^+} \left( \|\nabla Y(t)\|_{\dot{B}^{\frac{3}{2}}} + \|Y(t)\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}} + \|Y(t)\|_{\mathcal{B}_{2,\infty}^{4,-\frac{1}{2}}} \right. \\ \left. + \|\partial_3 Y(t)\|_{\dot{H}^{\frac{5}{2}}} + \|Y_t(t)\|_{\dot{H}^{\frac{5}{2}}} + \|\Delta Y(t)\|_{\dot{H}^{\frac{3}{4}}} \right) \leq \eta_0,$$

*and*

$$(5.2) \quad \lambda(t) \stackrel{\text{def}}{=} \|Y_t(t)\|_{\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}}} + \|Y(t)\|_{\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}}} + \|Y(t)\|_{\mathcal{B}_{2,\infty}^{\frac{5}{2},-\frac{1}{2}}} \leq \lambda_0$$

*for some  $\lambda_0 > 0$  and some sufficiently small  $\eta_0$ . Then one has*

$$(5.3) \quad \|Y_t(t)\|_{H^1}^2 + \|\partial_3 Y(t)\|_{H^1}^2 + \|Y(t)\|_{\dot{H}^2}^2 \lesssim \frac{(\lambda_0 + E_0)^2 E_0}{(\lambda_0 + E_0)^2 + E_0 \sqrt{t}}.$$

Let us remark that the proof of this proposition is motivated by similar ideas in [17, 26], which are formulated in the Eulerian coordinates. Moreover, compared with the result in [26], here we work out the limiting decay rate, namely, here the solution decays like  $\langle t \rangle^{-\frac{1}{4}}$ , while the solution in [26] decays like  $\langle t \rangle^{-s}$  for any  $s \in ]0, 1/4[$ .

*Proof.* We first get by a similar derivation of (4.4) that

$$\begin{aligned} \frac{d}{dt} \left( \frac{1}{2} (\|Y_t\|_{L^2}^2 + \|\partial_3 Y\|_{L^2}^2 + \frac{1}{4} \|\Delta Y\|_{L^2}^2) - \frac{1}{4} (Y_t | \Delta Y) \right) \\ + \frac{3}{4} \|\nabla Y_t\|_{L^2}^2 + \frac{1}{4} \|\nabla \partial_3 Y\|_{L^2}^2 = (f | (Y_t - \frac{1}{4} \Delta Y)). \end{aligned}$$

While by performing  $L^2$  inner product of (2.31) with  $-\Delta Y_t$ , we obtain

$$\frac{1}{2} \frac{d}{dt} (\|\nabla Y_t(t)\|_{L^2}^2 + \|\nabla \partial_3 Y(t)\|_{L^2}^2) + \|\Delta Y_t\|_{L^2}^2 = -(f | \Delta Y_t).$$

By summing up the above two inequalities, we obtain

$$(5.4) \quad \begin{aligned} \frac{d}{dt} \left( \frac{1}{2} (\|Y_t\|_{H^1}^2 + \|\partial_3 Y\|_{H^1}^2 + \frac{1}{4} \|\Delta Y\|_{L^2}^2) - \frac{1}{4} (Y_t | \Delta Y) \right) \\ + \frac{3}{4} \|\nabla Y_t\|_{L^2}^2 + \|\Delta Y_t\|_{L^2}^2 + \frac{1}{4} \|\nabla \partial_3 Y\|_{L^2}^2 \\ = (f | (Y_t - \frac{1}{4} \Delta Y - \Delta Y_t)), \end{aligned}$$

for  $f$  given by (2.32). Let us now deal with the last line of (5.4) term by term.

• The estimate of  $(\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t))(Y_t - \frac{1}{4}\Delta Y - \Delta Y_t)$

Due to

$$(5.5) \quad \mathcal{A}\mathcal{A}^t - Id = (\mathcal{A} - Id)(\mathcal{A} - Id)^t + \mathcal{A} - Id + (\mathcal{A} - Id)^t \quad \text{and} \quad \mathcal{A} - Id = \sum_{n=1}^{\infty} (\nabla Y)^n,$$

we get, by using the classical product law:

$$(5.6) \quad \|ab\|_{\dot{H}^s} \lesssim \|a\|_{\dot{B}^{\frac{3}{2}}} \|b\|_{\dot{H}^s} \quad \text{for } |s| < \frac{3}{2},$$

and (5.1) that

$$\begin{aligned} \|\mathcal{A}\mathcal{A}^t - Id\|_{\dot{B}^{\frac{3}{2}}} &\leq C(1 + \|\nabla Y\|_{L_t^\infty(\dot{B}^{\frac{3}{2}})}) \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} \\ &\leq C\|\nabla Y\|_{\dot{B}^{\frac{3}{2}}}, \end{aligned}$$

and

$$(5.7) \quad \begin{aligned} &|(\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t))(Y_t - \Delta Y_t)| \\ &\lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t\|_{L^2} \|\nabla Y_t\|_{L^2} + \|(\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t\|_{\dot{H}^1} \|\Delta Y_t\|_{L^2} \\ &\lesssim \|\mathcal{A}\mathcal{A}^t - Id\|_{\dot{B}^{\frac{3}{2}}} (\|\nabla Y_t\|_{L^2}^2 + \|\Delta Y_t\|_{L^2}^2) \\ &\lesssim \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} (\|\nabla Y_t\|_{L^2}^2 + \|\Delta Y_t\|_{L^2}^2). \end{aligned}$$

To deal with the term  $(\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t)|\Delta Y)$ , we write

$$(5.8) \quad \begin{aligned} &-(\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t)|\Delta Y) \\ &= -\frac{d}{dt} (\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y)|\Delta Y) + (\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y)|\Delta Y_t) \\ &\quad + (\nabla \cdot ((\partial_t((\mathcal{A} - Id)(\mathcal{A} - Id)^t) + \partial_t\mathcal{A} + \partial_t\mathcal{A}^t)\nabla Y)|\Delta Y). \end{aligned}$$

By virtue of (5.1) and Lemma 3.6, we deduce

$$\begin{aligned} |(\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y)|\Delta Y_t)| &\lesssim (1 + \|\nabla Y\|_{L^\infty}) \|\nabla Y\|_{L^4} \|\Delta Y\|_{L^4} \|\Delta Y_t\|_{L^2} \\ &\lesssim \|Y\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}}^{\frac{1}{2}} \|Y\|_{\mathcal{B}_{2,\infty}^{4,-\frac{1}{2}}}^{\frac{1}{2}} \|\nabla \partial_3 Y\|_{L^2} \|\Delta Y_t\|_{L^2}, \end{aligned}$$

and

$$\begin{aligned} &|(\nabla \cdot ((\partial_t((\mathcal{A} - Id)(\mathcal{A} - Id)^t) + \partial_t\mathcal{A} + \partial_t\mathcal{A}^t)\nabla Y)|\Delta Y)| \\ &\lesssim (1 + \|\nabla Y\|_{L^\infty}) (\|\nabla Y\|_{L^4} \|\Delta Y\|_{L^4} \|\Delta Y_t\|_{L^2} + \|\Delta Y\|_{L^4}^2 \|\nabla Y_t\|_{L^2}) \\ &\lesssim \left( \|Y\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}}^{\frac{1}{2}} \|Y\|_{\mathcal{B}_{2,\infty}^{4,-\frac{1}{2}}}^{\frac{1}{2}} \|\Delta Y_t\|_{L^2} + \|Y\|_{\mathcal{B}_{2,\infty}^{4,-\frac{1}{2}}} \|\nabla Y_t\|_{L^2} \right) \|\nabla \partial_3 Y\|_{L^2}. \end{aligned}$$

Hence, we obtain

$$(5.9) \quad \begin{aligned} &-(\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t)|\Delta Y) \lesssim -\frac{d}{dt} (\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y)|\Delta Y) \\ &\quad + \left( \|Y\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}} + \|Y\|_{\mathcal{B}_{2,\infty}^{4,-\frac{1}{2}}} \right) \|\nabla \partial_3 Y\|_{L^2} (\|\nabla Y_t\|_{L^2} + \|\Delta Y_t\|_{L^2}). \end{aligned}$$

•The estimate of  $(\mathcal{A}^t \nabla \mathbf{p})(-Y_t + \frac{1}{4}\Delta Y + \Delta Y_t)$

It is easy to observe that

$$\begin{aligned} &(\mathcal{A}^t \nabla \mathbf{p})(-Y_t + \frac{1}{4}\Delta Y + \Delta Y_t) \\ &= (\mathcal{A}^t \nabla \mathbf{p})(\frac{1}{4}\Delta Y + \Delta Y_t) - (\nabla \cdot (\mathcal{A}^t \mathbf{p})|Y_t) + (\nabla \cdot \mathcal{A}^t \mathbf{p}|Y_t) \\ &\lesssim (1 + \|\nabla Y\|_{L^\infty}) (\|\Delta Y\|_{L^2} + \|\Delta Y_t\|_{L^2}) \|\nabla \mathbf{p}\|_{L^2} \\ &\quad + \left( (1 + \|\nabla Y\|_{L^\infty}) \|\nabla Y_t\|_{L^2} + \|\Delta Y\|_{L^4} \|Y_t\|_{L^4} \right) \|\mathbf{p}\|_{L^2}. \end{aligned}$$

On the other hand, it follows from (2.32) that

$$\|\nabla \mathbf{p}\|_{L^2} \leq C \left( \|\nabla Y\|_{L^\infty} \|\nabla \mathbf{p}\|_{L^2} + \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{\dot{H}^1} \right),$$

so that as long as  $\eta_0$  in (5.1) is sufficiently small, we deduce from the product law (5.6) that

$$\begin{aligned}\|\nabla \mathbf{p}\|_{L^2} &\leq C(1 + \|\nabla Y\|_{L_t^\infty(B^{\frac{3}{2}})})\|(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{\dot{H}^1} \\ &\leq C(\|\partial_3 Y\|_{\dot{H}^{\frac{5}{4}}}^2 + \|Y_t\|_{\dot{H}^{\frac{5}{4}}}^2) \\ &\leq C(\|\nabla \partial_3 Y\|_{L^2}^{\frac{5}{3}} \|\partial_3 Y\|_{\dot{H}^{\frac{5}{2}}}^{\frac{1}{3}} + \|\nabla Y_t\|_{L^2}^{\frac{5}{3}} \|Y_t\|_{\dot{H}^{\frac{5}{2}}}^{\frac{1}{3}}).\end{aligned}$$

Along the same line, we deduce from (2.32) and the law of product (5.6) that

$$\begin{aligned}\|\mathbf{p}\|_{L^2} &\leq C\left(\|(\mathcal{A}\mathcal{A}^t - Id)\nabla \mathbf{p}\|_{\dot{H}^{-1}} + \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{L^2}\right) \\ &\leq C\left(\|\nabla Y\|_{B^{\frac{3}{2}}}\|\nabla \mathbf{p}\|_{\dot{H}^{-1}} + (1 + \|\nabla Y\|_{B^{\frac{3}{2}}})\|(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{L^2}\right),\end{aligned}$$

then under the assumption of (5.1), we have

$$\begin{aligned}\|\mathbf{p}\|_{L^2} &\leq C(\|\partial_3 Y\|_{L^4}^2 + \|Y_t\|_{L^4}^2) \\ &\leq C(\|\partial_3 Y\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 Y\|_{L^2}^{\frac{3}{2}} + \|Y_t\|_{L^2}^{\frac{1}{2}} \|\nabla Y_t\|_{L^2}^{\frac{3}{2}}).\end{aligned}$$

Therefore, by applying Lemmas 3.5 and 3.6, we arrive at

$$\begin{aligned}(5.10) \quad &|(\mathcal{A}^t \nabla \mathbf{p})(-Y_t + \frac{1}{4} \Delta Y + \Delta Y_t)| \\ &\lesssim \left(\|Y\|_{B_{2,\infty}^{\frac{5}{2}, -\frac{1}{2}}}^{\frac{2}{3}} \|\nabla \partial_3 Y\|_{L^2}^{\frac{1}{3}} + \|\Delta Y_t\|_{L^2}\right) \left(\|\nabla \partial_3 Y\|_{L^2}^{\frac{5}{3}} \|\partial_3 Y\|_{\dot{H}^{\frac{5}{2}}}^{\frac{1}{3}} + \|\nabla Y_t\|_{L^2}^{\frac{5}{3}} \|Y_t\|_{\dot{H}^{\frac{5}{2}}}^{\frac{1}{3}}\right) \\ &\quad + \left(\|\nabla Y_t\|_{L^2} + \|\Delta Y\|_{\dot{H}^{\frac{3}{4}}}\|Y_t\|_{L^2}^{\frac{1}{4}} \|\nabla Y_t\|_{L^2}^{\frac{3}{4}}\right) \left(\|\partial_3 Y\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 Y\|_{L^2}^{\frac{3}{2}} + \|Y_t\|_{L^2}^{\frac{1}{2}} \|\nabla Y_t\|_{L^2}^{\frac{3}{2}}\right).\end{aligned}$$

• The closure of the energy estimate

Let us denote

$$\begin{aligned}(5.11) \quad E_0(t) &\stackrel{\text{def}}{=} \frac{1}{2} \left( \|Y_t(t)\|_{H^1}^2 + \|\partial_3 Y(t)\|_{H^1}^2 + \frac{1}{4} \|\Delta Y(t)\|_{L^2}^2 \right) \\ &\quad - \frac{1}{4} (Y_t | \Delta Y) + (\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id) \nabla Y) | \Delta Y) \quad \text{and} \\ D_0(t) &\stackrel{\text{def}}{=} \|\nabla Y_t(t)\|_{H^1}^2 + \|\partial_3 \nabla Y(t)\|_{L^2}^2.\end{aligned}$$

Then by resuming the Estimates (5.7), (5.9) and (5.10) into (5.4), we obtain

$$\begin{aligned}\frac{d}{dt} E_0(t) + \frac{1}{4} D_0(t) &\leq C \left( \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} + \|Y\|_{B_{2,\infty}^{2, -\frac{1}{2}}} + \|Y\|_{B_{2,\infty}^{4, -\frac{1}{2}}} + \|\partial_3 Y\|_{\dot{H}^{\frac{5}{2}}} + \|Y_t\|_{\dot{H}^{\frac{5}{2}}} \right. \\ &\quad \left. + \|\partial_3 Y\|_{L^2} + \|Y_t\|_{L^2} + \|\Delta Y\|_{\dot{H}^{\frac{3}{4}}}\|Y_t\|_{L^2}^{\frac{1}{4}} \|\nabla Y_t\|_{L^2}^{\frac{1}{4}} (\|\partial_3 Y\|_{L^2}^{\frac{1}{2}} + \|Y_t\|_{L^2}^{\frac{1}{2}}) \right) D_0(t).\end{aligned}$$

Thus under the assumption of (5.1) and

$$(5.12) \quad C \sup_{t \in \mathbb{R}^+} (\|\partial_3 Y(t)\|_{L^2} + \|Y_t(t)\|_{L^2}) \leq \frac{1}{16},$$

we infer

$$(5.13) \quad \frac{d}{dt} E_0(t) + \frac{1}{8} D_0(t) \leq 0,$$

which in particular implies

$$(5.14) \quad E_0(t) \leq E_0 \quad \text{for } t \geq 0.$$

Note that  $\|\nabla Y\|_{L_t^\infty(L^\infty)} \leq \eta_0$ , we have

$$(5.15) \quad E_0(t) \sim \|Y_t(t)\|_{H^1}^2 + \|\partial_3 Y(t)\|_{H^1}^2 + \|\Delta Y(t)\|_{L^2}^2$$

Thus if  $\eta_0$  in (5.1) is sufficiently small, there holds (5.12) and (5.13) for all  $t \in \mathbb{R}^+$ .

On the other hand, it follows from Lemma 3.5 and (5.2) that

$$E_0(t) \leq C(\lambda(t) + \|\nabla Y_t(t)\|_{L^2} + \|\nabla \partial_3 Y(t)\|_{L^2})^{\frac{4}{3}} D_0(t)^{\frac{1}{3}} \leq C(\lambda_0 + E_0)^{\frac{4}{3}} D_0(t)^{\frac{1}{3}}.$$

Then we deduce from (5.13) that

$$\frac{d}{dt} E_0(t) + c(\lambda_0 + E_0)^{-4} E_0^3(t) \leq 0,$$

which together with (5.15) leads to (5.3). This completes the proof of Proposition 5.1.  $\square$

**Proposition 5.2.** *Under the assumptions of Proposition 5.1, if we assume moreover that*

$$(5.16) \quad \sup_{t \in \mathbb{R}^+} \left( \|\nabla Y(t)\|_{\dot{B}^{\frac{5}{2}}} + \|\nabla Y_t(t)\|_{H^2} + \|Y(t)\|_{\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}}} \right) \leq \eta_0, \quad \text{and} \quad \|\nabla Y\|_{L_t^\infty(\mathcal{B}_{2,\infty}^{\frac{5}{2},-\frac{1}{2}})} \leq C,$$

for some sufficiently small  $\eta_0$ . Then one has

$$(5.17) \quad \mathfrak{E}_1(t) \stackrel{\text{def}}{=} \|Y_t(t)\|_{H^2}^2 + \|\partial_3 Y(t)\|_{H^2}^2 + \|\Delta Y(t)\|_{H^1}^2 \lesssim \frac{\lambda_1^2 E_1}{\lambda_1^2 + E_1 \sqrt{t}}$$

with  $E_1 \stackrel{\text{def}}{=} \mathfrak{E}_1(0)$  and  $\lambda_1$  being given by

$$(5.18) \quad \lambda_1(t) \stackrel{\text{def}}{=} \lambda_0 + E_1 + \|\nabla Y(t)\|_{\mathcal{B}_{2,\infty}^{\frac{5}{2},-\frac{1}{2}}}^2 \leq \lambda_1.$$

*Proof.* We first get, by taking  $\partial_k$  to the System (2.31) and then taking the  $L^2$  inner product of the resulting equation with  $\partial_k Y_t - \frac{1}{4} \Delta \partial_k Y - \Delta \partial_k Y_t$ , that

$$(5.19) \quad \begin{aligned} & \frac{d}{dt} \left( \frac{1}{2} (\|\partial_k Y_t\|_{H^1}^2 + \|\partial_k \partial_3 Y\|_{H^1}^2 + \frac{1}{4} \|\partial_k Y\|_{H^2}^2) - \frac{1}{4} (\partial_k Y_t | \Delta \partial_k Y) \right) \\ & + \frac{3}{4} \|\nabla \partial_k Y_t\|_{L^2}^2 + \|\Delta \partial_k Y_t\|_{L^2}^2 + \frac{1}{4} \|\nabla \partial_k \partial_3 Y\|_{L^2}^2 \\ & = (\partial_k f | (\partial_k Y_t - \frac{1}{4} \Delta \partial_k Y - \Delta \partial_k Y_t)) \quad \text{for } k = 1, 2, 3. \end{aligned}$$

We now deal with the last line of (5.19) term by term. It follows from the classical product law, (5.6), that

$$(5.20) \quad \begin{aligned} & |(\nabla \cdot \partial_k ((\mathcal{A}\mathcal{A}^t - Id) \nabla Y_t) | (\partial_k Y_t - \Delta \partial_k Y_t))| \\ & \lesssim \|\partial_k ((\mathcal{A}\mathcal{A}^t - Id) \nabla Y_t)\|_{\dot{H}^1} (\|\partial_k Y_t\|_{L^2} + \|\Delta \partial_k Y_t\|_{L^2}) \\ & \lesssim (\|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} \|\Delta \partial_k Y_t\|_{L^2} + \|\nabla \partial_k Y\|_{\dot{B}^{\frac{3}{2}}} \|\Delta Y_t\|_{L^2}) (\|\partial_k Y_t\|_{L^2} + \|\Delta \partial_k Y_t\|_{L^2}) \\ & \lesssim (\|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} + \|\nabla \partial_k Y\|_{\dot{B}^{\frac{3}{2}}}) (\|\partial_k Y_t\|_{L^2}^2 + \|\Delta Y_t\|_{L^2}^2 + \|\Delta \partial_k Y_t\|_{L^2}^2). \end{aligned}$$

Similar to (5.8), one has

$$\begin{aligned} & -(\nabla \cdot \partial_k ((\mathcal{A}\mathcal{A}^t - Id) \nabla Y_t) | \Delta \partial_k Y) \\ & = -\frac{d}{dt} (\nabla \cdot \partial_k ((\mathcal{A}\mathcal{A}^t - Id) \nabla Y) | \Delta \partial_k Y) + (\nabla \cdot \partial_k ((\mathcal{A}\mathcal{A}^t - Id) \nabla Y) | \Delta \partial_k Y_t) \\ & \quad + (\nabla \cdot \partial_k ((\partial_t ((\mathcal{A} - Id)(\mathcal{A} - Id)^t) + \partial_t \mathcal{A} + \partial_t \mathcal{A}^t) \nabla Y) | \Delta \partial_k Y). \end{aligned}$$

It follows from Lemma 3.6 that

$$\begin{aligned} & |(\nabla \cdot \partial_k ((\mathcal{A}\mathcal{A}^t - Id) \nabla Y) | \Delta \partial_k Y_t)| \\ & \lesssim \|\partial_k ((\mathcal{A}\mathcal{A}^t - Id) \nabla Y)\|_{\dot{H}^1} \|\Delta \partial_k Y_t\|_{L^2} \\ & \lesssim (1 + \|\nabla Y\|_{L^\infty}) (\|\nabla Y\|_{L^4} \|\nabla^2 \partial_k Y\|_{L^4} + \|\nabla^2 Y\|_{L^4}^2) \|\Delta \partial_k Y_t\|_{L^2} \\ & \lesssim (\|Y\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}} + \|Y\|_{\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}}}) (\|\nabla \partial_3 Y\|_{H^1}^2 + \|\Delta \partial_k Y_t\|_{L^2}^2), \end{aligned}$$



and

$$\begin{aligned}
& |(\nabla \cdot \partial_k((\mathcal{A} - Id)(\mathcal{A} - Id)^t) + \partial_t \mathcal{A} + \partial_t \mathcal{A}^t) \nabla Y)| \Delta \partial_k Y| \\
& \lesssim (1 + \|\nabla Y\|_{L^\infty}) (\|\nabla Y\|_{L^4} \|\Delta \partial_k Y_t\|_{L^2} + \|\nabla^2 Y\|_{L^4} \|\Delta Y_t\|_{L^2} + \|\nabla Y_t\|_{L^2} \|\Delta \partial_k Y\|_{L^4}) \|\Delta \partial_k Y\|_{L^4} \\
& \lesssim (\|Y\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}} + \|Y\|_{\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}}}) (\|\nabla \partial_3 Y\|_{H^1}^2 + \|\nabla Y_t\|_{H^2}^2).
\end{aligned}$$

This gives

$$\begin{aligned}
(5.21) \quad & -(\nabla \cdot \partial_k((\mathcal{A} \mathcal{A}^t - Id) \nabla Y_t)| \Delta \partial_k Y) \\
& = -\frac{d}{dt} (\nabla \cdot \partial_k((\mathcal{A} \mathcal{A}^t - Id) \nabla Y)| \Delta \partial_k Y) \\
& \quad + (\|Y\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}} + \|Y\|_{\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}}}) (\|\nabla \partial_3 Y\|_{H^1}^2 + \|\nabla Y_t\|_{H^2}^2).
\end{aligned}$$

While we deduce from the classical product law, (5.6), that

$$\begin{aligned}
& |(\partial_k(\mathcal{A}^t \nabla \mathbf{p})|(-\partial_k Y_t + \frac{1}{4} \Delta \partial_k Y + \Delta \partial_k Y_t))| \\
& \lesssim \|\mathcal{A}^t \nabla \mathbf{p}\|_{\dot{H}^1} (\|\nabla Y_t\|_{H^2} + \|\Delta \partial_k Y\|_{L^2}) \\
& \lesssim (1 + \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}}) \|\nabla \mathbf{p}\|_{\dot{H}^1} (\|\nabla Y_t\|_{H^2} + \|\Delta \partial_k Y\|_{L^2}).
\end{aligned}$$

On the other hand, we infer from (2.32) and the law of product, (5.6), that

$$\begin{aligned}
\|\nabla \mathbf{p}\|_{\dot{H}^1} & \lesssim \|(\mathcal{A} \mathcal{A}^t - Id) \nabla \mathbf{p}\|_{\dot{H}^1} + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t))\|_{\dot{H}^1} \\
& \lesssim \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} \|\nabla \mathbf{p}\|_{\dot{H}^1} + (1 + \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}}) \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{\dot{H}^2},
\end{aligned}$$

which together with, (5.1), (5.16) and the interpolation inequality:  $\|a\|_{\dot{B}^{\frac{3}{2}}} \lesssim \|a\|_{\dot{H}^1}^{\frac{1}{2}} \|a\|_{\dot{H}^2}^{\frac{1}{2}}$ , ensures that

$$\begin{aligned}
(5.22) \quad & \|\nabla \mathbf{p}\|_{\dot{H}^1} \lesssim (1 + \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} + \|\nabla Y\|_{\dot{B}^{\frac{5}{2}}}) (\|\partial_3 Y\|_{\dot{B}^{\frac{3}{2}}} \|\nabla \partial_3 Y\|_{H^1} + \|Y_t\|_{\dot{B}^{\frac{3}{2}}} \|\nabla Y_t\|_{H^1}) \\
& \lesssim \|\nabla \partial_3 Y\|_{H^1}^2 + \|\nabla Y_t\|_{H^1}^2.
\end{aligned}$$

Hence under the assumptions of (5.1) and (5.16), we obtain

$$(5.23) \quad |(\partial_k(\mathcal{A}^t \nabla \mathbf{p})|(-\partial_k Y_t + \frac{1}{4} \Delta \partial_k Y + \Delta \partial_k Y_t))| \lesssim (\|\nabla Y\|_{\dot{H}^2} + \|\nabla Y_t\|_{H^2}) (\|\nabla \partial_3 Y\|_{H^1}^2 + \|\nabla Y_t\|_{H^2}^2).$$

Let us now denote

$$\begin{aligned}
(5.24) \quad & \dot{E}_1(t) \stackrel{\text{def}}{=} \frac{1}{2} (\|\nabla Y_t(t)\|_{H^1}^2 + \|\nabla \partial_3 Y(t)\|_{H^1}^2 + \frac{1}{4} \|\nabla Y(t)\|_{H^2}^2) \\
& \quad - \frac{1}{4} (\nabla Y_t \mid \nabla \Delta Y) + \sum_{k=1}^3 (\nabla \cdot \partial_k((\mathcal{A} \mathcal{A}^T - Id) \nabla Y)| \Delta \partial_k Y), \\
& \dot{D}_1(t) \stackrel{\text{def}}{=} \|\Delta Y_t(t)\|_{H^1}^2 + \|\partial_3 Y(t)\|_{H^2}^2, \quad \text{and} \\
& E_1(t) \stackrel{\text{def}}{=} E_0(t) + \dot{E}_1(t), \quad D_1(t) \stackrel{\text{def}}{=} D_0(t) + \dot{D}_1(t).
\end{aligned}$$

Note that

$$\sum_{k=1}^3 |(\nabla \cdot \partial_k((\mathcal{A} \mathcal{A}^T - Id) \nabla Y)| \Delta \partial_k Y)| \lesssim \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} \|\nabla Y\|_{H^2}^2,$$

so that there holds

$$\begin{aligned}
(5.25) \quad & \dot{E}_1(t) \sim \|\nabla Y_t(t)\|_{H^1}^2 + \|\nabla \partial_3 Y(t)\|_{H^1}^2 + \|\nabla Y(t)\|_{H^2}^2, \\
& E_1(t) \sim \|Y_t(t)\|_{H^2}^2 + \|\partial_3 Y(t)\|_{H^2}^2 + \|\Delta Y(t)\|_{H^1}^2.
\end{aligned}$$

Then by resuming the inequalities (5.20), (5.21) and (5.23) into (5.19), we obtain

$$(5.26) \quad \frac{d}{dt}\dot{E}_1(t) + \frac{1}{4}\dot{D}_1(t) \leq C \left( \|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} + \|\nabla Y\|_{\dot{B}^{\frac{5}{2}}} + \|\nabla Y_t\|_{H^2} + \|Y\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}} + \|Y\|_{\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}}} \right) D_1(t).$$

By summing up (5.13) with (5.26) and using the smallness assumptions (5.1) and (5.16) leads to

$$(5.27) \quad \frac{d}{dt}E_1(t) + \frac{1}{16}D_1(t) \leq 0,$$

which in particular implies

$$(5.28) \quad E_1(t) + \frac{1}{16} \int_s^t D_1(t') dt' \leq E_1(s) \quad \forall s \in [0, t].$$

In particular (5.25) and (5.28) ensures that

$$(5.29) \quad \|Y_t(t)\|_{H^2}^2 + \|\partial_3 Y(t)\|_{H^2}^2 + \|\Delta Y(t)\|_{H^1}^2 + \int_0^t (\|\nabla Y_t(t')\|_{H^2}^2 + \|\nabla \partial_3 Y(t')\|_{H^1}^2) dt' \leq CE_1.$$

Then we deduce from Lemma 3.5 and (5.25) that

$$\begin{aligned} E_1(t) &\leq C(\lambda_0 + E_0 + \|\nabla Y_t(t)\|_{H^1}^2 + \|\nabla \partial_3 Y(t)\|_{H^1}^2 + \|\nabla Y(t)\|_{\mathcal{B}_{2,\infty}^{\frac{5}{2},-\frac{1}{2}}}^2)^{\frac{2}{3}} D_1^{\frac{1}{3}}(t) \\ &\leq C(\lambda_0 + E_1 + \|\nabla Y(t)\|_{\mathcal{B}_{2,\infty}^{\frac{5}{2},-\frac{1}{2}}}^2)^{\frac{2}{3}} D_1^{\frac{1}{3}}(t) \\ &\leq C\lambda_1^{\frac{4}{3}} D_1^{\frac{1}{3}}(t), \end{aligned}$$

which together with (5.27) ensures that

$$\frac{d}{dt}E_1(t) + c\lambda_1^{-4}E_1^3(t) \leq 0,$$

which leads to (5.17). This completes the proof of Proposition 5.2.  $\square$

**Proposition 5.3.** *Under the assumptions of Proposition 5.2, one has*

$$(5.30) \quad \|\partial_3 Y_t(t)\|_{H^1}^2 + \|\partial_3^2 Y(t)\|_{H^1}^2 + \|\partial_3 Y(t)\|_{\dot{H}^2}^2 \lesssim \langle t \rangle^{-\frac{3}{4}}.$$

*Proof.* It follows from the classical product law, (5.6), that

$$\begin{aligned} (5.31) \quad &|(\nabla \cdot \partial_3((\mathcal{A}\mathcal{A}^T - Id)\nabla Y_t) \mid \Delta \partial_3 Y)| \\ &\leq \|\partial_3((\mathcal{A}\mathcal{A}^T - Id)\nabla Y_t)\|_{\dot{H}^1} \|\partial_3 Y\|_{\dot{H}^2} \\ &\lesssim (\|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} \|\partial_3 Y_t\|_{\dot{H}^2} + \|\nabla Y_t\|_{\dot{B}^{\frac{3}{2}}} \|\partial_3 Y\|_{\dot{H}^2}) \|\partial_3 Y\|_{\dot{H}^2}. \end{aligned}$$

We denote

$$\begin{aligned} \dot{E}_1^3(t) &\stackrel{\text{def}}{=} \frac{1}{2} (\|\partial_3 Y_t(t)\|_{H^1}^2 + \|\partial_3^2 Y(t)\|_{H^1}^2 + \frac{1}{4} \|\partial_3 Y(t)\|_{\dot{H}^2}^2) - \frac{1}{4} (\partial_3 Y_t \mid \partial_3 \Delta Y), \quad \text{and} \\ \dot{D}_1^3(t) &\stackrel{\text{def}}{=} \|\nabla \partial_3 Y_t(t)\|_{H^1}^2 + \|\partial_3^2 Y(t)\|_{H^1}^2. \end{aligned}$$

Then resuming the Inequalities (5.20), (5.23) and (5.31) into (5.19) for  $k = 3$  gives rise to

$$\begin{aligned} \frac{d}{dt}\dot{E}_1^3(t) + \frac{1}{4}\dot{D}_1^3(t) &\leq C(\|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} + \|\nabla Y\|_{\dot{H}^2} + \|\nabla Y_t\|_{H^2}) \\ &\quad \times (\|\partial_3 Y_t\|_{H^2}^2 + \|\nabla Y_t\|_{H^2}^2 + \|\nabla \partial_3 Y\|_{H^1}^2), \end{aligned}$$

from which and the smallness condition and (5.1), (5.16), we infer

$$\begin{aligned} & \frac{d}{dt} \dot{E}_1^3(t) + \frac{1}{8} \dot{D}_1^3(t) \\ & \leq C(\|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} + \|\nabla Y\|_{\dot{H}^2} + \|\nabla Y_t\|_{H^2})(\|\partial_3 Y_t\|_{L^2}^2 + \|\nabla Y_t\|_{H^2}^2 + \|\nabla \partial_3 Y\|_{H^1}^2) \\ & \leq C(\|\nabla Y\|_{\dot{B}^{\frac{3}{2}}} + \|\nabla Y\|_{\dot{B}^{\frac{5}{2}}} + \|\nabla Y_t\|_{H^2}) D_1(t) \leq C\eta_0 D_1(t). \end{aligned}$$

which together with (5.28) yields

$$(5.32) \quad \dot{E}_1^3(t) + \int_0^t \dot{D}_1^3(t') dt' \leq \dot{E}_1^3(0) + C\eta_0 \int_0^t D_1(t') dt' \leq C(\dot{E}_1^3(0) + E_1).$$

Moreover, note that  $\dot{E}_1^3(t) \lesssim D_1(t)$ , for any  $0 < s \leq t$ , we have

$$\begin{aligned} & \frac{d}{dt}((t-s)\dot{E}_1^3(t)) + (t-s)\dot{D}_1^3(t) \leq \dot{E}_1^3(t) + C(t-s)(\|\nabla Y\|_{\dot{H}^1} + \|\nabla Y\|_{\dot{H}^2} + \|\nabla Y_t\|_{H^1}) D_1(t) \\ & \leq C\left(1 + (t-s)(\|\nabla Y\|_{\dot{H}^1} + \|\nabla Y\|_{\dot{H}^2} + \|\nabla Y_t\|_{H^1})\right) D_1(t). \end{aligned}$$

Then in view of (5.3), (5.17) and (5.28), we get, by integrating the above inequality over  $[s, t]$  and then taking  $s = \frac{t}{2}$ , that

$$\begin{aligned} t\dot{E}_1^3(t) & \leq C\left(\int_{\frac{t}{2}}^t D_1(t') dt' + t \int_{\frac{t}{2}}^t (\|\nabla Y(t')\|_{\dot{H}^1} + \|\nabla Y(t')\|_{\dot{H}^2} + \|\nabla Y_t(t')\|_{H^1}) D_1(t') dt'\right) \\ & \leq C\left(\dot{E}_1(t/2) + \langle t \rangle^{\frac{3}{4}} \int_{\frac{t}{2}}^t D_1(t') dt'\right) \\ & \leq C\langle t \rangle^{\frac{3}{4}} \dot{E}_1(t/2) \leq C\langle t \rangle^{\frac{1}{4}}, \end{aligned}$$

which together with (5.32) leads to (5.30). This completes the proof of Proposition 5.3.  $\square$

## 6. PROPAGATION OF REGULARITIES IN THE LAGRANGIAN COORDINATE

In this section, we prove the regularity estimates, which are required by the last section, namely, (5.1), (5.2) and (5.16).

**Proposition 6.1.** *Let  $s > -1$  and  $Y$  be a smooth enough solution of (2.31) on  $[0, T]$ , which satisfies the Inequality (4.11). We denote*

$$\begin{aligned} E_s(t) & \stackrel{\text{def}}{=} \|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}^{s,0})} + \|\partial_3 Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{s,0})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+2,0})} \\ & \quad + \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}^{s+1,0})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}^{s+1,0})} + \|Y_t\|_{L_t^1(\mathcal{B}^{s+2,0})} + \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{s,0})}. \end{aligned}$$

Then under the assumption of (2.33), we have

$$(6.1) \quad E_s(t) \lesssim c_0 + \|\partial_3 Y_0\|_{\mathcal{B}^{s,0}} + \|Y_0\|_{\mathcal{B}^{s+2,0}} + \|Y_1\|_{\mathcal{B}^{s,0}} \quad \text{for } t \in ]0, T].$$

*Proof.* In view of (2.31), we get, by a similar derivation (4.1), that

$$\begin{aligned} & \|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}^{s,0})} + \|\partial_3 Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{s,0})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{s+2,0})} \\ & \quad + \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}^{s+1,0})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}^{s+1,0})} + \|Y_t\|_{L_t^1(\mathcal{B}^{s+2,0})} \\ & \lesssim \|\partial_3 Y_0\|_{\mathcal{B}^{s,0}} + \|Y_0\|_{\mathcal{B}^{s+2,0}} + \|Y_1\|_{\mathcal{B}^{s,0}} + \|f\|_{L_t^1(\mathcal{B}^{s,0})}. \end{aligned} \quad (6.2)$$

Let us now handle term by term of  $\|f\|_{L_t^1(\mathcal{B}^{s,0})}$  for  $f$  given by (2.32). It follows from the law of product, (3.6), that

$$\begin{aligned} \|\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t)\|_{L_t^1(\mathcal{B}^{s,0})} & \lesssim \|\mathcal{A}\mathcal{A}^t - Id\|_{L_t^\infty(\mathcal{B}^{s+1,0})} \|\nabla Y_t\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})} \\ & \quad + \|\mathcal{A}\mathcal{A}^t - Id\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})} \|Y_t\|_{L_t^1(\mathcal{B}^{s+2,0})}. \end{aligned}$$

Note from (4.11) that

$$\|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})} \leq C\|Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{2,\frac{1}{2}})} \leq Cc_0,$$

and thus by the law of product, Lemma 3.3, (5.5) and (4.11), we have

$$(6.3) \quad \begin{aligned} \|\mathcal{A}\mathcal{A}^t - Id\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})} &\leq C\|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})} \leq Cc_0, \quad \text{and} \\ \|\mathcal{A}\mathcal{A}^t - Id\|_{L_t^\infty(\mathcal{B}^{s+1,0})} &\leq C(1 + \|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})})\|Y\|_{L_t^\infty(\mathcal{B}^{s+2,0})} \leq C\|Y\|_{L_t^\infty(\mathcal{B}^{s+2,0})}. \end{aligned}$$

So that by virtue of (4.11), we infer

$$(6.4) \quad \begin{aligned} \|\nabla \cdot ((\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t)\|_{L_t^1(\mathcal{B}^{s,0})} &\leq C(\|Y\|_{L_t^\infty(\mathcal{B}^{s+2,0})}\|\nabla Y_t\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})} + c_0\|Y_t\|_{L_t^1(\mathcal{B}^{s+2,0})}) \\ &\leq Cc_0(\|Y\|_{L_t^\infty(\mathcal{B}^{s+2,0})} + \|Y_t\|_{L_t^1(\mathcal{B}^{s+2,0})}). \end{aligned}$$

Similarly we deduce from (2.32), (3.6) and (4.11) that

$$\begin{aligned} \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{s,0})} &\lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{s,0})} + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t))\|_{L_t^1(\mathcal{B}^{s,0})} \\ &\lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}\|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{s,0})} + \|(\mathcal{A}\mathcal{A}^T - Id)\|_{\tilde{L}_t^\infty(\mathcal{B}^{1+s,0})}\|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\ &\quad + (1 + \|\mathcal{A} - Id\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})})\|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{L_t^1(\mathcal{B}^{1+s,0})} \\ &\quad + \|\nabla \mathcal{A}\|_{L_t^\infty(\mathcal{B}^{1+s,0})}\|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})}. \end{aligned}$$

Yet it follows from (4.11) and the law of product, Lemma 3.3, that

$$\begin{aligned} \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{L_t^1(\mathcal{B}^{1,\frac{1}{2}})} \\ \lesssim (1 + \|\mathcal{A} - Id\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}) (\|\partial_3 Y\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}^2 + \|Y_t\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}^2) \leq Cc_0^2, \end{aligned}$$

and

$$\begin{aligned} \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{L_t^1(\mathcal{B}^{1+s,0})} \\ \lesssim (1 + \|\mathcal{A} - Id\|_{L_t^\infty(\mathcal{B}^{1+s,0})}) (\|\partial_3 Y\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}^2 + \|Y_t\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}^2) \\ + (1 + \|\mathcal{A} - Id\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}) (\|\partial_3 Y\|_{L_t^2(\mathcal{B}^{1,\frac{1}{2}})}\|\partial_3 Y\|_{L_t^2(\mathcal{B}^{1+s,0})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}^{1,\frac{1}{2}})}\|Y_t\|_{L_t^2(\mathcal{B}^{1+s,0})}) \\ \leq Cc_0(c_0 + \|Y\|_{L_t^\infty(\mathcal{B}^{2+s,0})} + \|\partial_3 Y\|_{L_t^2(\mathcal{B}^{1+s,0})} + \|Y_t\|_{L_t^2(\mathcal{B}^{1+s,0})}). \end{aligned}$$

Hence in view of (4.11) and (6.3), we get

$$(6.5) \quad \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{s,0})} \leq Cc_0(c_0 + \|Y\|_{L_t^\infty(\mathcal{B}^{2+s,0})} + \|\partial_3 Y\|_{L_t^2(\mathcal{B}^{1+s,0})} + \|Y_t\|_{L_t^2(\mathcal{B}^{1+s,0})}).$$

Therefore thanks to (3.6) and (4.11), we have

$$\begin{aligned} \|\mathcal{A}^t \nabla \mathbf{p}\|_{\tilde{L}_t^1(\mathcal{B}^{s,0})} &\lesssim (1 + \|\mathcal{A} - Id\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})})\|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{s,0})} + (1 + \|\nabla \mathcal{A}\|_{L_t^\infty(\mathcal{B}^{s,0})})\|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,\frac{1}{2}})} \\ &\leq Cc_0(c_0 + \|Y\|_{L_t^\infty(\mathcal{B}^{2+s,0})} + \|\partial_3 Y\|_{L_t^2(\mathcal{B}^{1+s,0})} + \|Y_t\|_{L_t^2(\mathcal{B}^{1+s,0})}), \end{aligned}$$

which together with (6.4) ensures that

$$\|f\|_{L_t^1(\mathcal{B}^{s,0})} \leq Cc_0(c_0 + \|Y\|_{L_t^\infty(\mathcal{B}^{s+2,0})} + \|\partial_3 Y\|_{L_t^2(\mathcal{B}^{s+1,0})} + \|Y_t\|_{L_t^2(\mathcal{B}^{s+1,0})} + \|Y_t\|_{L_t^1(\mathcal{B}^{s+2,0})}).$$

Then by resuming the above estimate into (6.2) and taking  $c_0$  to be sufficiently small gives rise to

$$\begin{aligned} &\|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}^{s,0})} + \|\partial_3 Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{s,0})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{2+s,0})} + \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}^{s+1,0})} \\ &\quad + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}^{s+1,0})}^2 + \|Y_t\|_{L_t^1(\mathcal{B}^{s+2,0})} \\ &\leq C(c_0 + \|\partial_3 Y_0\|_{\mathcal{B}^{s,0}} + \|Y_0\|_{\mathcal{B}^{s+2,0}} + \|Y_1\|_{\mathcal{B}^{s,0}}), \end{aligned}$$

from which and (6.5), we deduce (6.1). This completes the proof of the Proposition.  $\square$

An immediate corollary of Proposition 6.1 and Definitions 1.1 and 1.2 gives

**Corollary 6.1.** *Under the assumptions of Proposition 6.1, one has*

$$(6.6) \quad \begin{aligned} & \|Y_t\|_{\tilde{L}_T^\infty(\dot{H}^s)} + \|\partial_3 Y\|_{\tilde{L}_T^\infty(\dot{H}^s)} + \|Y\|_{\tilde{L}_T^\infty(\dot{H}^{2+s})} + \|Y_t\|_{L_T^2(\dot{H}^{s+1})} \\ & + \|\partial_3 Y\|_{L_T^2(\dot{H}^{s+1})} + \|Y_t\|_{L_T^1(\dot{H}^{2+s})} + \|\nabla \mathbf{p}\|_{L_t^1(\dot{H}^s)} \\ & \leq C \left( c_0 + \|\partial_3 Y_0\|_{\mathcal{B}^{s,0}} + \|Y_0\|_{\mathcal{B}^{s+2,0}} + \|Y_1\|_{\mathcal{B}^{s,0}} \right) \quad \text{for any } s > -1. \end{aligned}$$

**Proposition 6.2.** *Let  $Y$  be a smooth enough solution of (2.31) on  $[0, T]$ , which satisfies the Estimate (4.11). Then under the assumptions of (2.33) and (2.35), One has*

$$(6.7) \quad \|Y_t\|_{L_t^\infty(\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}})} + \|Y_t\|_{L_t^\infty(\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}})} + \|Y\|_{L_t^\infty(\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}})} + \|Y\|_{L_t^\infty(\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}})} + \|Y\|_{L_t^\infty(\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}})} \leq C(c_0 + \delta_0)$$

for any  $t \leq T$ .

*Proof.* We first deduce from Proposition 6.1 and (2.35) that

$$(6.8) \quad E_0(t) + E_3(t) \leq C(c_0 + \delta_0)$$

for  $E_s(t)$  given by Proposition 6.1. While in view of Definition 1.2, we get, by a similar derivation of (4.1), that for all  $s \in \mathbb{R}$ ,

$$(6.9) \quad \|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{s,-\frac{1}{2}})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{2+s,-\frac{1}{2}})} \lesssim \|\partial_3 Y_0\|_{\mathcal{B}_{2,\infty}^{s,-\frac{1}{2}}} + \|Y_0\|_{\mathcal{B}_{2,\infty}^{2+s,-\frac{1}{2}}} + \|Y_1\|_{\mathcal{B}_{2,\infty}^{s,-\frac{1}{2}}} + \|f\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{s,-\frac{1}{2}})}.$$

• **The estimate of  $\|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}})}$  and  $\|Y\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}})}$**

It is easy to observe from Definition 1.2 that

$$\|a\|_{\mathcal{B}_{2,\infty}^{s,-\frac{1}{2}}} \lesssim \|a\|_{\mathcal{B}_{\infty,\infty}^{s_1,-\frac{1}{2}}}^\theta \|a\|_{\mathcal{B}_{2,\infty}^{s_2,-\frac{1}{2}}}^{1-\theta} \quad \text{with } s = \theta s_1 + (1-\theta)s_2 \quad \text{and } \theta \in ]0, 1[ ,$$

from which and (2.35), we infer

$$(6.10) \quad \|\partial_3 Y_0\|_{\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}}} + \|Y_0\|_{\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}}} + \|Y_1\|_{\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}}} \leq C\delta_0.$$

Then thanks to (6.9), we only need to deal with the estimate of  $\|f\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}})}$ . Indeed according to

(2.32), we deduce from Lemma 3.4 that

$$(6.11) \quad \begin{aligned} \|f\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}})} & \lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{1,-\frac{1}{2}})} + \|\mathcal{A}^t \nabla \mathbf{p}\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}})} \\ & \lesssim \|\mathcal{A}\mathcal{A}^t - Id\|_{L_t^\infty(\mathcal{B}^{1,0})} \|Y_t\|_{L_t^1(\mathcal{B}^{2,0})} + (1 + \|\mathcal{A} - Id\|_{L^\infty(\mathcal{B}^{1,0})}) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,0})} \\ & \lesssim (1 + \|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}) \|Y\|_{L_t^\infty(\mathcal{B}^{2,0})} \|Y_t\|_{L_t^1(\mathcal{B}^{2,0})} \\ & \quad + \left( 1 + (1 + \|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}) \|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,0})} \right) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,0})}. \end{aligned}$$

Yet it follows from (2.32) and the law of product, Lemma 3.3, that

$$\begin{aligned} \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,0})} & \lesssim \|(\mathcal{A}\mathcal{A}^T - Id)\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,0})} + \|\mathcal{A} \operatorname{div}(\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t))\|_{L_t^1(\mathcal{B}^{0,0})} \\ & \lesssim \|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})} \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,0})} \\ & \quad + (1 + \|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}) \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{L_t^1(\mathcal{B}^{1,0})}, \end{aligned}$$

from which, (4.11), Lemma 3.3 and Lemma 3.1, we infer

$$\begin{aligned}
\|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,0})} &\lesssim \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{L_t^1(\mathcal{B}^{1,0})} \\
&\lesssim (1 + \|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}) (\|\partial_3 Y\|_{L_t^2(\mathcal{B}^{1,\frac{1}{4}}})^2 + \|Y_t\|_{L_t^2(\mathcal{B}^{1,\frac{1}{4}}})^2) \\
(6.12) \quad &\lesssim \|\partial_3 Y\|_{L_t^2(\dot{B}^{\frac{5}{4}})}^2 + \|Y_t\|_{L_t^2(\dot{B}^{\frac{5}{4}})}^2 \\
&\lesssim \|\partial_3 Y\|_{L_t^2(\mathcal{B}^{\frac{5}{4},0})}^2 + \|Y_t\|_{L_t^2(\mathcal{B}^{\frac{5}{4},0})}^2.
\end{aligned}$$

Resuming the Estimate (6.12) into (6.11) and using (6.8), we obtain

$$\begin{aligned}
\|f\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}})} &\lesssim \|Y\|_{L_t^\infty(\mathcal{B}^{2,0})} \|Y_t\|_{L_t^1(\mathcal{B}^{2,0})} + (c_0 + \delta_0)(1 + \|Y\|_{L_t^\infty(\mathcal{B}^{2,0})}) \\
(6.13) \quad &\leq C(c_0 + \delta_0).
\end{aligned}$$

Thus in view of (6.10) and (6.9), we conclude

$$(6.14) \quad \|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{0,-\frac{1}{2}})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{2,-\frac{1}{2}})} \leq C(c_0 + \delta_0).$$

• **The estimate of  $\|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}})}$  and  $\|Y\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}})}$**

In view of (2.32), we get, by applying the law of product Lemma 3.4, that

$$\begin{aligned}
\|f\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}})} &\lesssim \|(\mathcal{A}^t - Id)\nabla Y_t\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{4,-\frac{1}{2}})} + \|\mathcal{A}^t \nabla \mathbf{p}\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}})} \\
&\lesssim \|\mathcal{A}^T - Id\|_{L^\infty(\mathcal{B}^{4,0})} \|Y_t\|_{L_t^1(\mathcal{B}^{2,0})} + \|\mathcal{A}^T - Id\|_{L^\infty(\mathcal{B}^{1,0})} \|Y_t\|_{L_t^1(\mathcal{B}^{5,0})} \\
&\quad + (1 + \|\mathcal{A} - Id\|_{L^\infty(\mathcal{B}^{1,0})}) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{3,0})} + (1 + \|\mathcal{A} - Id\|_{L^\infty(\mathcal{B}^{4,0})}) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,0})} \\
&\lesssim \|Y\|_{L^\infty(\mathcal{B}^{2,0})} \|Y_t\|_{L_t^1(\mathcal{B}^{5,0})} + (1 + \|Y\|_{L^\infty(\mathcal{B}^{2,0})}) \|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{3,0})} \\
&\quad + (1 + \|Y\|_{L^\infty(\mathcal{B}^{5,0})}) (\|\nabla \mathbf{p}\|_{L_t^1(\mathcal{B}^{0,0})} + \|Y_t\|_{L_t^1(\mathcal{B}^{2,0})}),
\end{aligned}$$

from which, (6.8) and (6.12), we infer

$$\|f\|_{\tilde{L}_t^1(\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}})} \leq C(c_0 + \delta_0).$$

While due to (2.35), one has

$$\|\partial_3 Y_0\|_{\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}}} + \|Y_0\|_{\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}}} + \|Y_1\|_{\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}}} \leq C\delta_0.$$

Thus we deduce from (6.9) that

$$(6.15) \quad \|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{3,-\frac{1}{2}})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}_{2,\infty}^{5,-\frac{1}{2}})} \leq C(c_0 + \delta_0).$$

• **The estimate of  $\|Y\|_{\tilde{L}_t^\infty(\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}})}$**

Again in view of (2.31), we get, by a similar derivation of (4.1), that

$$\begin{aligned}
\|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}})} &+ \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + \|Y_t\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}})} \\
(6.16) \quad &\lesssim \|\partial_3 Y_0\|_{\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}}} + \|Y_0\|_{\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}}} + \|Y_1\|_{\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}}} + \|f\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})}.
\end{aligned}$$

To deal with the estimate  $\|f\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})}$ , we deduce from (2.32), the law of product, Lemma 3.4, that

$$\begin{aligned} \|f\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})} &\lesssim \|(\mathcal{A}\mathcal{A}^t - Id)\nabla Y_t\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + (1 + \|\nabla Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{1,\frac{1}{2}})})\|\nabla p\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})} \\ &\lesssim \|\mathcal{A}\mathcal{A}^t - Id\|_{\tilde{L}_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}\|\nabla Y_t\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + \|\nabla p\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})} \\ &\lesssim \|\nabla Y\|_{\tilde{L}_t^\infty(\mathcal{B}^{1,\frac{1}{2}})}\|Y_t\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}})} + \|\nabla p\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})}. \end{aligned}$$

While it follows from (2.32), Lemma 3.3 and (4.11) that

$$\begin{aligned} \|\nabla p\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})} &\lesssim \|\mathcal{A}(\partial_3 Y \otimes \partial_3 Y - Y_t \otimes Y_t)\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} \\ &\lesssim \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}^{1,\frac{1}{2}})}\|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}^{1,\frac{1}{2}})}\|Y_t\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} \\ &\leq Cc_0\left(\|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})}\right). \end{aligned}$$

Therefore, we achieve

$$\|f\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})} \leq Cc_0\left(\|Y_t\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}})} + \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})}\right).$$

Resuming the above estimate into (6.16) and using (2.35) gives rise to

$$(6.17) \quad \|Y_t\|_{\tilde{L}_t^\infty(\mathcal{B}_{\infty,\infty}^{-1,-\frac{1}{2}})} + \|Y\|_{\tilde{L}_t^\infty(\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}})} + \|\partial_3 Y\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + \|Y_t\|_{\tilde{L}_t^2(\mathcal{B}_{\infty,\infty}^{0,-\frac{1}{2}})} + \|Y_t\|_{\tilde{L}_t^1(\mathcal{B}_{\infty,\infty}^{1,-\frac{1}{2}})} \leq C\delta_0.$$

Finally it follows from Definition 3.1 that

$$\|a\|_{L_t^\infty(\mathcal{B}_{p,\infty}^{s,-\frac{1}{2}})} \leq \|a\|_{\tilde{L}_t^\infty(\mathcal{B}_{p,\infty}^{s,-\frac{1}{2}})} \quad \text{for any } p \in [1, \infty].$$

Then by summing up the Estimates (6.14), (6.15) and (6.17), we conclude the proof of (6.7).  $\square$

## 7. THE PROOF OF THEOREMS 1.1 AND 2.1

Let us first present the proof of Theorem 2.1.

*Proof of Theorem 2.1.* In general, the existence of solutions to a nonlinear partial differential equation can be obtained by performing uniform estimates to the appropriate approximate solutions. Here for simplicity, we just present the *a priori* estimate for smooth enough solution,  $Y$ , of (2.31). Indeed under the assumption of (2.33), we first deduce from Proposition 4.1 that  $Y$  satisfies the Inequality (4.11). Then it follows from Proposition 6.1 that there holds (2.34), which ensures the global existence part of Theorem 2.1. The uniqueness of such smooth solution is standard, we omit the details here.

In order to prove the decay estimate (2.36), we need to verify the smallness conditions (5.1) and (5.16), which are guaranteed by (2.34) and Proposition 6.2 provided that there holds (2.35). This completes the proof of Theorem 2.1.  $\square$

Now we are in a position to complete the proof of Theorem 1.1. Let us first recall Proposition 6.1 from [29]:

**Proposition 7.1.** *Let  $b_0 - e_3 \in H^s(\mathbb{R}^3)$  and  $u_0 \in H^s(\mathbb{R}^3)$  for  $s > \frac{3}{2}$ , (1.1) has a unique solution  $(b, u)$  on  $[0, T]$  for some  $T > 0$  so that  $b - e_3 \in C([0, T]; H^s(\mathbb{R}^3))$ ,  $u \in C([0, T]; H^s(\mathbb{R}^3))$  with  $\nabla u \in L^2((0, T); H^s(\mathbb{R}^3))$  and  $\nabla p \in C([0, T]; H^{s-1}(\mathbb{R}^3))$ . Moreover, if  $T^*$  is the life span to this solution, and  $T^* < \infty$ , one has*

$$(7.1) \quad \int_0^{T^*} (\|\nabla u(t)\|_{L^\infty} + \|b(t)\|_{L^\infty}^2) dt = \infty.$$

*Proof of Theorem 1.1.* Given initial data  $(b_0, u_0)$  which satisfies the assumptions of Theorem 1.1, we deduce from Proposition 7.1 that (1.1) has a unique solution  $(b, u)$  on  $[0, T^*[$  such that for any  $T < T^*$ ,

$$b - e_3 \in C([0, T]; H^s(\mathbb{R}^3)), \quad u \in C([0, T]; H^s(\mathbb{R}^3)) \quad \text{with} \quad \nabla u \in L^2((0, T); H^s(\mathbb{R}^3)).$$

Moreover, it follows from the transport equation of (1.1) that

$$\|b(t)\|_{L^\infty} \leq \|b_0\|_{L^\infty} \exp\left(\|\nabla u\|_{L_t^1(L^\infty)}\right).$$

Therefore, by virtue of Proposition 7.1, in order to complete the existence part of Theorem 1.1, it remains to prove that

$$(7.2) \quad \int_0^{T^*} \|\nabla u(t)\|_{L^\infty} dt < \infty.$$

Toward this, we introduce the equivalent Lagrangian formulation (2.12), which has been presented in details in Section 2. Indeed, according to the derivation in Section 2, especially (2.15) and (2.16), one has

$$(7.3) \quad \begin{aligned} Y_1(z) &= u_0(y_h(z_h, w_3(z)), w_3(z)) \quad \text{and} \quad Y_t(t, y) = u(t, y + Y(t, y)) \quad \text{with} \\ Y(t, (y_h(z_h, w_3(z)), w_3(z))) &= \tilde{Y}(z) + \bar{Y}(t, z), \end{aligned}$$

with  $\tilde{Y}(z)$  and  $\bar{Y}(t, z)$  being determined by (4.19) and (2.24) respectively.

On the other hand, let us recall (A.3) of [29] that

$$(7.4) \quad \|u \circ \Phi\|_{\dot{B}_{p,r}^s} \leq C(\|\nabla \Psi\|_{L^\infty}) \|u\|_{\dot{B}_{p,r}^s} \quad \text{for } s \in ]-1, 1[.$$

While it follows from (2.19), (A.3) and (A.7) that

$$(7.5) \quad \|\mathcal{B}\|_{L^\infty} \leq \left\| \frac{\partial z}{\partial w} \right\|_{L^\infty} \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty}^{-1} \leq 2 \quad \text{for } \varepsilon \leq \varepsilon_0.$$

Thus by virtue of (2.21), Lemma 3.1 and (7.4), we infer

$$\|Y_1\|_{\mathcal{B}^{0, \frac{1}{2}}} \leq C(\|\mathcal{B}\|_{L^\infty}) \|u_0\|_{\dot{B}^{\frac{1}{2}}} \leq C \|u_0\|_{\dot{B}^{\frac{1}{2}}}.$$

So that under the assumption of (1.6), we deduce from (4.20) and Proposition 4.1 that for any  $t < T^*$  and  $\varepsilon \leq \varepsilon_0$ ,

$$(7.6) \quad \|\nabla \bar{Y}\|_{L_t^\infty(L^\infty)} + \|\nabla \tilde{Y}\|_{L_t^1(L^\infty)} \lesssim \|\nabla \bar{Y}\|_{\tilde{L}_t^\infty(\mathcal{B}^{1, \frac{1}{2}})} + \|\bar{Y}_t\|_{L_t^1(\mathcal{B}^{2, \frac{1}{2}})} \leq C c_0.$$

Hence it follows from (2.11) and (7.3) that for any  $T < T^*$

$$\begin{aligned} \int_0^T \|\nabla u(t)\|_{L^\infty} dt &\leq \int_0^T \|\mathcal{A}^t \nabla_y Y_t(t)\|_{L^\infty} dt \leq \int_0^T \|\mathcal{A}^t \mathcal{B}^t \nabla_z \bar{Y}_t(t)\|_{L^\infty} dt \\ &\leq (1 + \|\nabla Y\|_{L_t^\infty(\mathcal{B}^{1, \frac{1}{2}})} + \|\nabla \tilde{Y}\|_{\mathcal{B}^{1, \frac{1}{2}}}) \|\mathcal{B}\|_{L_t^\infty(L^\infty)} \int_0^T \|\nabla_z \bar{Y}_t(t)\|_{L^\infty} dt \\ &\leq C \int_0^T \|\nabla_z \bar{Y}_t(t)\|_{L^\infty} dt \leq C c_0. \end{aligned}$$

This proves (7.2) and thus the global existence part of Theorem 1.1 is proved.

In the case when  $b_0 = e_3$ , by virtue of (2.6), (2.8) and (2.11), we find that  $Y(t, y)$  determined by (2.11) solves the System (2.31). Moreover, there holds

$$(7.7) \quad Y_0 = 0, \quad b(t, X(t, y)) = e_3 + \partial_3 Y(t, y) \quad \text{and} \quad u(t, X(t, x)) = Y_t(t, y).$$

Then under the assumptions of (1.6) and (1.7), there hold the Inequalities (2.33) and (2.35) so that by virtue of Theorem 2.1, (2.31) has a unique global solution  $Y$  which satisfies (2.34) and (2.36).



On the other hand, due to  $\operatorname{div} u = 0$ , we have

$$\begin{aligned} \|b(t, \cdot) - e_3\|_{L^2} &= \|b(t, X(t, \cdot)) - e_3\|_{L^2} = \|\partial_3 Y(t)\|_{L^2}, \\ \|\nabla b(t, \cdot)\|_{L^2} &= \|\mathcal{A}^t \nabla b(t, X(t, \cdot))\|_{L^2} = \|\mathcal{A}^t \nabla \partial_3 Y(t)\|_{L^2} \lesssim \|\nabla \partial_3 Y(t)\|_{L^2}, \end{aligned}$$

and

$$\begin{aligned} \|\nabla^2 b(t, \cdot)\|_{L^2} &= \|(\mathcal{A}^t \nabla)^2 \partial_3 Y(t)\|_{L^2} \\ &\leq C(1 + \|\nabla Y\|_{L_t^\infty(L^\infty)} + \|\nabla^2 Y\|_{L_t^\infty(L^\infty)}) \|\nabla \partial_3 Y(t)\|_{H^1} \\ &\leq C(1 + \|\nabla Y\|_{L_t^\infty(B^{1, \frac{1}{2}})} + \|\nabla^2 Y\|_{L_t^\infty(B^{\frac{3}{2}, 0})}) \|\nabla \partial_3 Y(t)\|_{H^1} \leq C \|\nabla \partial_3 Y(t)\|_{H^1}. \end{aligned}$$

Exactly along the same line, one has

$$\begin{aligned} \|u(t, \cdot)\|_{L^2} &= \|u(t, X(t, \cdot))\|_{L^2} = \|Y_t(t)\|_{L^2}, \\ \|\nabla u(t, \cdot)\|_{L^2} &= \|\mathcal{A}^t \nabla u(t, X(t, \cdot))\|_{L^2} = \|\mathcal{A}^t \nabla Y_t(t)\|_{L^2} \lesssim \|\nabla Y_t(t)\|_{L^2}, \\ \|\nabla^2 u(t, \cdot)\|_{L^2} &= \|(\mathcal{A}^t \nabla)^2 Y_t(t)\|_{L^2} \lesssim \|\nabla Y_t(t)\|_{H^1}, \end{aligned}$$

which together with (2.36) ensures (1.8). This completes the proof of Theorem 1.1.  $\square$

#### APPENDIX A. THE PROOF OF LEMMA 4.2

In this section, we always denote  $\phi = (\phi_h, \phi_3)$ ,  $y = (y_h, y_3)$  and  $z = (z_h, z_3)$ . The proof of Lemma 4.2 will be based on the following two lemmas.

**Lemma A.1.** *Let  $y(w)$  be determined by (2.15) and  $g(z_h) \stackrel{\text{def}}{=} \int_0^K G_1(y_h(z_h, y_3), y_3) dy_3$ . Then for  $\varepsilon \leq \varepsilon_\alpha$ , which depends only on  $\|\nabla \phi\|_{W^{2, \infty}}$ , one has*

$$(A.1) \quad \|g_1\|_{H_h^2} \leq CK \left( \sum_{|\alpha| \leq 2} \|\nabla_h^\alpha G_1\|_{L_v^\infty(L_h^2)} \right) \quad \text{and} \quad \|\nabla_h g_1\|_{H_h^2} \leq CK \left( \sum_{|\alpha| \leq 2} \|\nabla_h^\alpha \nabla_h G_1\|_{L_v^\infty(L_h^2)} \right).$$

*Proof.* We first deduce from (2.18) that

$$(A.2) \quad \|A_2\|_{L^\infty} \leq \varepsilon K \left\| \nabla_h \left( \frac{\phi_h}{1 + \varepsilon \phi_3} \right) \right\|_{L^\infty} \leq 3K\varepsilon \|\nabla_h \phi\|_{L^\infty} \leq \frac{1}{2},$$

whenever  $\varepsilon \leq \varepsilon_a \stackrel{\text{def}}{=} \min\left(\frac{1}{4\|\phi_3\|_{L^\infty}}, \frac{1}{6K\|\nabla_h \phi\|_{L^\infty}}\right)$ . Then by virtue of (2.19), we have

$$(A.3) \quad \frac{1}{3} \leq \frac{\|A_1\|_{L^\infty}}{1 + \|A_2\|_{L^\infty}} \leq \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty} \leq \frac{\|A_1\|_{L^\infty}}{1 - \|A_2\|_{L^\infty}} \leq 2.$$

Similarly, we deduce from (2.18) that

$$\begin{aligned} (A.4) \quad \left\| \frac{\partial^2 y}{\partial w^2} \right\|_{L^\infty} &\leq \|(Id - A_2)^{-1}\|_{L^\infty} \left( \left\| \frac{\partial A_1}{\partial y} \right\|_{L^\infty} + \left\| \frac{\partial A_2}{\partial y} \right\|_{L^\infty} \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty} \right) \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty} \\ &\leq C\varepsilon \sum_{|\alpha| \leq 1} \|\nabla^\alpha \nabla \phi\|_{L^\infty} \leq 1, \end{aligned}$$

and

$$\begin{aligned} (A.5) \quad \left\| \frac{\partial^3 y}{\partial w^3} \right\|_{L^\infty} &\leq C \|(Id - A_2)^{-1}\|_{L^\infty} \left( \left( \left\| \frac{\partial^2 A_1}{\partial y^2} \right\|_{L^\infty} + \left\| \frac{\partial^2 A_2}{\partial y^2} \right\|_{L^\infty} \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty} \right) \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty} \right. \\ &\quad \left. + \left( \left\| \frac{\partial A_1}{\partial y} \right\|_{L^\infty} + \left\| \frac{\partial A_2}{\partial y} \right\|_{L^\infty} \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty} \right) \left\| \frac{\partial^2 y}{\partial w^2} \right\|_{L^\infty} \right) \\ &\leq C\varepsilon \sum_{|\alpha| \leq 2} \|\nabla^\alpha \nabla \phi\|_{L^\infty} \leq 1, \end{aligned}$$

provided that  $\varepsilon \leq \varepsilon_\alpha \stackrel{\text{def}}{=} \min(\varepsilon_a, C^{-1}(\sum_{|\alpha| \leq 2} \|\nabla^\alpha \nabla \phi\|_{L^\infty})^{-1})$ .

Then thanks to (A.3), one has

$$\|g_1\|_{L_h^2} \leq CK \|G_1\|_{L_v^\infty(L_h^2)}.$$

While for  $k, \ell, m \in \{1, 2\}$ , we have

$$\frac{\partial G_1(y_h(z_h, y_3), y_3)}{\partial z_k} = \sum_{i=1}^2 \frac{\partial G_1}{\partial y_i}(y_h(z_h, y_3), y_3) \frac{\partial y_i(z_h, y_3)}{\partial z_k},$$

and

$$\frac{\partial^2 G_1(y_h(z_h, y_3), y_3)}{\partial z_\ell \partial z_k} = \sum_{i=1}^2 \frac{\partial G_1}{\partial y_i}(y_h(z_h, y_3), y_3) \frac{\partial^2 y_i}{\partial z_\ell \partial z_k} + \sum_{i,j=1}^2 \frac{\partial^2 G_1}{\partial y_j \partial y_i}(y_h(z_h, y_3), y_3) \frac{\partial y_i}{\partial z_k} \frac{\partial y_j}{\partial z_\ell},$$

and

$$\begin{aligned} \frac{\partial^3 G_1(y_h(z_h, y_3), y_3)}{\partial z_m \partial z_\ell \partial z_k} &= \sum_{i=1}^2 \frac{\partial G_1}{\partial y_i}(y_h(z_h, y_3), y_3) \frac{\partial^3 y_i}{\partial z_m \partial z_\ell \partial z_k} \\ &+ \sum_{i,j=1}^2 \frac{\partial^2 G_1}{\partial y_j \partial y_i}(y_h(z_h, y_3), y_3) \left( \frac{\partial^2 y_i}{\partial z_\ell \partial z_k} \frac{\partial y_j}{\partial z_m} + \frac{\partial^2 y_i}{\partial z_k \partial z_m} \frac{\partial y_j}{\partial z_\ell} \right. \\ &\left. + \frac{\partial y_i}{\partial z_k} \frac{\partial^2 y_j}{\partial z_m \partial z_\ell} \right) + \sum_{i,j,l=1}^2 \frac{\partial^3 G_1}{\partial y_l \partial y_j \partial y_i}(y_h(z_h, y_3), y_3) \frac{\partial y_l}{\partial z_m} \frac{\partial y_i}{\partial z_k} \frac{\partial y_j}{\partial z_\ell}, \end{aligned}$$

from which, (A.3), (A.4) and (A.5), we infer

$$\|\nabla_h g_1\|_{L_h^2} \leq CK \|\nabla_h G_1\|_{L_v^\infty(L_h^2)},$$

and

$$\begin{aligned} \|\nabla_h^2 g_1\|_{L_h^2} &\leq C \int_0^K \left( \|\nabla_h^2 G_1(y_h(\cdot, y_3), y_3)\|_{L_h^2} \|\nabla_h y_h\|_{L^\infty}^2 \right. \\ &\quad \left. + \|\nabla_h G_1(y_h(\cdot, y_3), y_3)\|_{L_h^2} \|\nabla_h^2 y_h\|_{L^\infty} \right) dy_3 \\ &\leq CK \left( \|\nabla_h G_1\|_{L_v^\infty(L_h^2)} + \|\nabla_h^2 G_1\|_{L_v^\infty(L_h^2)} \right). \end{aligned}$$

Similarly, one has

$$\|\nabla_h^3 g_1\|_{L_h^2} \leq CK \sum_{|\alpha| \leq 2} \|\nabla_h^\alpha \nabla_h G_1\|_{L_v^\infty(L_h^2)}.$$

As a consequence, we achieve (A.1). □

**Lemma A.2.** *Let  $y(w)$  and  $w(z)$  be determined respectively by (2.15) and (2.16), let  $g_2(z) \stackrel{\text{def}}{=} G_2(y(w(z)))$ . Then for  $\varepsilon \leq \varepsilon_\beta$ , which depends only on  $\|\nabla \phi\|_{W^{2,\infty}}$ , one has*

$$(A.6) \quad \|g_2\|_{B^{1,\frac{1}{2}}} \leq C \|g_2\|_{\dot{B}^{\frac{3}{2}}} \leq C \|\nabla G_2\|_{L^2}^{\frac{1}{2}} \|\nabla G_2\|_{H^1}^{\frac{1}{2}}.$$

*Proof.* It follows from (2.14) and (2.20) that

$$(A.7) \quad \begin{aligned} \left\| \frac{\partial z}{\partial w} - Id \right\|_{L^\infty} &\leq C \int_0^K \left\| \nabla_h \left( \frac{1}{1 + \varepsilon \phi_3} \right) (y_h(\cdot, w_3), w_3) \right\|_{L^\infty} \left\| \frac{\partial y_h}{\partial w_h} \right\|_{L^\infty} dw_3 + \left\| \frac{\varepsilon \phi_3}{1 + \varepsilon \phi_3} \right\|_{L^\infty} \\ &\leq C \varepsilon (\|\phi_3\|_{L^\infty} + \|\nabla_h \phi_3\|_{L^\infty}) \leq \frac{1}{2}, \end{aligned}$$

provided that  $\varepsilon \leq \varepsilon_b \stackrel{\text{def}}{=} \min(\varepsilon_\alpha, (2C)^{-1} (\|\phi_3\|_{L^\infty} + \|\nabla_h \phi_3\|_{L^\infty})^{-1})$  for  $\varepsilon_\alpha$  given by Lemma A.1.

While by virtue of (2.16) and (2.21), we have

$$z_3 = w_3(z) - \varepsilon \int_0^{w_3(z)} \Phi_\varepsilon(y_h(z_h, y_3), y_3) dy_3,$$

where  $\Phi_\varepsilon(y) \stackrel{\text{def}}{=} \frac{\phi_3(y)}{1+\varepsilon\phi_3(y)}$ . Then due to (2.21), for  $k, \ell \in \{1, 2\}$ , we have

$$(1 - \varepsilon \Phi_\varepsilon(y_h(z_h, w_3(z)), w_3(z))) \frac{\partial w_3(z)}{\partial z_k} = \varepsilon \int_0^{w_3(z)} \nabla_h \Phi_\varepsilon(y_h(z_h, y_3), y_3) \cdot \frac{\partial y_h}{\partial z_k}(z_h, y_3) dy_3,$$

and

$$\begin{aligned} & (1 - \varepsilon \Phi_\varepsilon(y(w(z)))) \frac{\partial^2 w_3}{\partial z_k \partial z_\ell} - \varepsilon \frac{\partial \Phi_\varepsilon}{\partial y_3}(y(w(z))) \frac{\partial w_3}{\partial z_k} \frac{\partial w_3}{\partial z_\ell} \\ & - \varepsilon \nabla_h \Phi_\varepsilon(y(w(z))) \cdot \left( \frac{\partial y_h}{\partial z_\ell}(z_h, w_3(z)) + \frac{\partial y_h}{\partial w_3}(z_h, w_3(z)) \frac{\partial w_3}{\partial z_\ell} \right) \frac{\partial w_3}{\partial z_k} \\ & = \varepsilon \nabla_h \Phi_\varepsilon(y(w(z))) \cdot \frac{\partial y_h}{\partial z_k}(z_h, w_3(z)) \frac{\partial w_3}{\partial z_\ell} + \varepsilon \int_0^{w_3(z)} \nabla_h \Phi_\varepsilon(y_h(z_h, y_3), y_3) \cdot \frac{\partial^2 y_h}{\partial z_\ell \partial z_k}(z_h, y_3) dy_3 \\ & + \varepsilon \sum_{i,j=1}^2 \int_0^{w_3(z)} \frac{\partial^2 \Phi_\varepsilon}{\partial y_i \partial y_j}(y_h(z_h, y_3), y_3) \frac{\partial y_j}{\partial z_\ell}(z_h, y_3) \frac{\partial y_i}{\partial z_k}(z_h, y_3) dy_3. \end{aligned}$$

Note that  $\varepsilon \|\phi_3\|_{L^\infty} \leq \frac{1}{4}$ , for  $k, \ell \in \{1, 2\}$ , we have

$$\begin{aligned} \left\| \frac{\partial^2 w_3}{\partial z_k \partial z_\ell} \right\|_{L^\infty} & \leq C\varepsilon \left( \|\nabla \Phi_\varepsilon\|_{L^\infty} \left( \left\| \frac{\partial w_3}{\partial z} \right\|_{L^\infty} + \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty} + \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty} \left\| \frac{\partial w_3}{\partial z} \right\|_{L^\infty} \right) \left\| \frac{\partial w_3}{\partial z} \right\|_{L^\infty} \right. \\ & \quad \left. + K \|\nabla \Phi\|_{L^\infty} \left\| \frac{\partial^2 y}{\partial w^2} \right\|_{L^\infty} + K \|\nabla^2 \Phi\|_{L^\infty} \left\| \frac{\partial y}{\partial w} \right\|_{L^\infty}^2 \right). \end{aligned}$$

so that by virtue of (A.3) and (A.4), we infer

$$(A.8) \quad \left\| \frac{\partial^2 w_3}{\partial z_k \partial z_\ell} \right\|_{L^\infty} \leq C\varepsilon (\|\nabla \phi_3\|_{L^\infty} + \|\nabla^2 \phi_3\|_{L^\infty}) \leq 1,$$

provided that  $\varepsilon \leq \varepsilon_\beta \stackrel{\text{def}}{=} \min(\varepsilon_b, C^{-1}(\|\nabla \phi_3\|_{L^\infty} + \|\nabla^2 \phi_3\|_{L^\infty})^{-1})$ . Similar calculation shows that (A.8) holds for all  $k, \ell \in \{1, 2, 3\}$ .

On the other hand, for  $k, \ell \in \{1, 2\}$ , one has

$$\begin{aligned} \frac{\partial g_2(z)}{\partial z_k} & = \frac{\partial G_2}{\partial y_3}(y_h(z_h, w_3(z)), w_3(z)) \frac{\partial w_3(z)}{\partial z_k} \\ & + \sum_{i=1}^2 \frac{\partial G_2}{\partial y_i}(y_h(z_h, w_3(z)), w_3(z)) \left( \frac{\partial y_i}{\partial z_k} + \frac{\partial y_i}{\partial w_3} \frac{\partial w_3(z)}{\partial z_k} \right) (z_h, w_3(z)), \end{aligned}$$

and

$$\begin{aligned}
\frac{\partial^2 g_2(z)}{\partial z_k \partial z_\ell} &= \frac{\partial^2 G_2}{\partial y_3^2}(y(w(z))) \frac{\partial w_3}{\partial z_\ell} \frac{\partial w_3}{\partial z_k} + \frac{\partial G_2}{\partial y_3}(y(w(z))) \frac{\partial^2 w_3}{\partial z_\ell \partial z_k} \\
&+ \sum_{j=1}^2 \frac{\partial^2 G_2}{\partial y_3 \partial y_j}(y(w(z))) \left( \frac{\partial y_j}{\partial z_\ell}(w(z)) + \frac{\partial y_j}{\partial w_3}(w(z)) \frac{\partial w_3}{\partial z_\ell} \right) \\
&+ \sum_{i,j=1}^2 \frac{\partial^2 G_2}{\partial y_i \partial y_j}(y(w(z))) \left( \frac{\partial y_j}{\partial z_\ell} + \frac{\partial y_j}{\partial w_3} \frac{\partial w_3}{\partial z_\ell} \right) \left( \frac{\partial y_i}{\partial z_k} + \frac{\partial y_i}{\partial w_3} \frac{\partial w_3}{\partial z_k} \right) (w(z)) \\
&+ \sum_{i=1}^2 \frac{\partial G_2}{\partial y_i}(y(w(z)), w_3(z)) \left( \frac{\partial^2 y_i}{\partial z_k \partial z_\ell} + \frac{\partial^2 y_i}{\partial w_3 \partial z_k} \frac{\partial w_3}{\partial z_\ell} \right. \\
&\quad \left. + \frac{\partial^2 y_i}{\partial w_3 \partial z_\ell} \frac{\partial w_3}{\partial z_k} + \frac{\partial^2 y_i}{\partial w_3^2} \frac{\partial w_3}{\partial z_k} \frac{\partial w_3}{\partial z_\ell} + \frac{\partial y_i}{\partial w_3} \frac{\partial^2 w_3}{\partial z_k \partial z_\ell} \right) (w(z)),
\end{aligned}$$

from which, (A.3), (A.7) and (A.8), we deduce that

$$\begin{aligned}
\|\nabla_h g_2\|_{L^2} &\leq C \|\nabla G_2\|_{L^2} (1 + \|\frac{\partial y}{\partial w}\|_{L^\infty}) (1 + \|\frac{\partial w_3}{\partial z}\|_{L^\infty}) \|\det(\frac{\partial y}{\partial w})\|_{L^\infty} \|\det(\frac{\partial w}{\partial z})\|_{L^\infty} \\
&\leq C \|\nabla G_2\|_{L^2},
\end{aligned}$$

and

$$\begin{aligned}
\|\nabla_h^2 g_2\|_{L^2} &\leq C \|\nabla^2 G_2\|_{L^2} \left( \|\frac{\partial y}{\partial w}\|_{L^\infty} + \|\frac{\partial w_3}{\partial z}\|_{L^\infty}^2 + \|\frac{\partial y}{\partial w}\|_{L^\infty}^2 (1 + \|\frac{\partial w_3}{\partial z}\|_{L^\infty}^2) \right) \\
&\quad + C \|\nabla G_2\|_{L^2} \left( (1 + \|\frac{\partial y}{\partial w}\|_{L^\infty}) \|\frac{\partial^2 w_3}{\partial z^2}\|_{L^\infty} + (1 + \|\frac{\partial w_3}{\partial z}\|_{L^\infty}^2) \|\frac{\partial^2 y}{\partial w^2}\|_{L^\infty} \right) \\
&\leq C (\|\nabla G_2\|_{L^2} + \|\nabla^2 G_2\|_{L^2}).
\end{aligned}$$

Similar calculation shows that the above two estimates hold for the full derivatives of  $g_2$ . Hence by virtue of Lemma 3.1 and the interpolation inequality in Besov space, we deduce (A.6). This finishes the proof of the lemma.  $\square$

Let us now turn to the proof of Lemmas 4.2 and 4.3.

*Proof of Lemma 4.2.* By virtue of (A.1), for  $A_2^h$  given by (4.15), we get, by using interpolation inequality in Besov spaces, that for  $\varepsilon \leq \varepsilon_\alpha$ ,

$$(A.9) \quad \|A_2^h\|_{\dot{B}_h^1} \leq C \|A_2^h\|_{H_h^2} \leq C \varepsilon \left( \sum_{|\alpha| \leq 2} \|\nabla_h^\alpha \nabla_h \phi\|_{L_v^\infty(L_h^2)} \right).$$

Similarly by applying the second equality of (A.1), we deduce that,  $\|A_3^h\|_{\dot{B}_h^1}$ , for  $A_3^h$  given by (4.16), satisfies the same estimate as  $\|A_2^h\|_{\dot{B}_h^1}$ .

Note that for  $b_0 = e_3 + \varepsilon \phi$ , we deduce from (2.18) and (A.6) that for  $\varepsilon \leq \varepsilon_\beta$

$$(A.10) \quad \|\mathfrak{A}_1 - Id\|_{\mathcal{B}^{1, \frac{1}{2}}} \leq C \varepsilon \|\nabla \phi\|_{L^2}^{\frac{1}{2}} \|\nabla \phi\|_{H^1}^{\frac{1}{2}}.$$

Due to (4.15), (4.16) and the fact that  $\eta$  is supported on  $[-2, K+2]$ , we get, by a similar derivation of (A.6), that

$$(A.11) \quad \|A_{2,1}\|_{\mathcal{B}^{1, \frac{1}{2}}} + \|A_{3,1}\|_{\mathcal{B}^{1, \frac{1}{2}}} \leq C \varepsilon \|\nabla \phi\|_{H^2}.$$

Let us take

$$\varepsilon_2 \stackrel{\text{def}}{=} \min \left( \varepsilon_\alpha, \varepsilon_\beta, C^{-1} (\|\nabla \phi\|_{H^2} + \|\nabla_h \phi\|_{L_v^\infty(H_h^2)})^{-1} \right).$$

Then (A.11) together with (A.9) and (A.10) leads to (4.18).  $\square$

*Proof of Lemma 4.3.* Let us denote

$$\mathfrak{Y}(z) \stackrel{\text{def}}{=} \varepsilon \eta(z_3) \int_{-1}^{z_3} \phi(y_h(z_h, w_3(z')), w_3(z')) dz'_3 \quad \text{with} \quad z' = (z_h, z'_3).$$

Then one has

$$\frac{\partial \mathfrak{Y}(z)}{\partial z_3} = \varepsilon \eta'(z_3) \int_{-1}^{z_3} \phi(y_h(z_h, w_3(z_h, z'_3)), w_3(z_h, z'_3)) dz'_3 + \varepsilon \eta(z_3) \phi(y(w(z))),$$

and

$$\begin{aligned} \frac{\partial^2 \mathfrak{Y}(z)}{\partial^2 z_3} &= \varepsilon \eta''(z_3) \int_{-1}^{z_3} \phi(y(w(z'))) dz'_3 + 2\varepsilon \eta'(z_3) \phi(y(w(z))) \\ &\quad + \varepsilon \eta(z_3) \left( \frac{\partial \phi}{\partial y_h}(y(w(z))) \cdot \frac{\partial y_h}{\partial w_3}(w(z)) \frac{\partial w_3}{\partial z_3}(z) + \frac{\partial \phi}{\partial y_3}(y(w(z))) \frac{\partial w_3}{\partial z_3}(z) \right), \end{aligned}$$

from which and (A.3), we infer

$$(A.12) \quad \|\partial_3 \mathfrak{Y}\|_{L^2} \leq C\varepsilon (\|\phi\|_{L^2} + \|\phi\|_{L^\infty_\nabla(L_h^2)}).$$

Similarly, for  $k = 1, 2$ , one has

$$\begin{aligned} \frac{\partial^2 \mathfrak{Y}(z)}{\partial z_3 \partial z_k} &= \varepsilon \eta'(z_3) \int_{-1}^{z_3} \left( \frac{\partial \phi}{\partial y_h}(y(w(z'))) \cdot \left( \frac{\partial y_h}{\partial z_k}(z_h, w(z')) + \frac{\partial y_h}{\partial w_3}(z_h, w(z')) \frac{\partial w_3}{\partial z_k}(z') \right) \right. \\ &\quad \left. + \frac{\partial \phi}{\partial y_3}(y(w(z'))) \frac{\partial w_3}{\partial z_k}(z') \right) dz'_3 + \varepsilon \eta(z_3) \left( \frac{\partial \phi}{\partial y_3}(y(w(z))) \frac{\partial w_3}{\partial z_k}(z) \right. \\ &\quad \left. + \frac{\partial \phi}{\partial y_h}(y(w(z))) \cdot \left( \frac{\partial y_h}{\partial z_k}(z_h, w(z)) + \frac{\partial y_h}{\partial w_3}(z_h, w(z)) \frac{\partial w_3}{\partial z_k}(z) \right) \right). \end{aligned}$$

So that we obtain

$$\begin{aligned} (A.13) \quad \|\nabla \partial_3 \mathfrak{Y}\|_{L^2} &\leq C\varepsilon \left( \|\phi\|_{L^2} + \|\phi\|_{L^\infty_\nabla(L_h^2)} + (\|\nabla \phi\|_{L^2} + \|\nabla \phi\|_{L^\infty_\nabla(L_h^2)}) \right. \\ &\quad \left. \times (1 + \|\frac{\partial y}{\partial w}\|_{L^\infty}) (1 + \|\frac{\partial w}{\partial z}\|_{L^\infty}) \right) \\ &\leq C\varepsilon (\|\phi\|_{H^1} + \|\phi\|_{L^\infty_\nabla(H_h^1)}). \end{aligned}$$

Then by virtue of Lemma 3.1 and the classical interpolation inequality in Besov spaces (see [2]), we deduce from (A.12) and (A.13) that

$$\begin{aligned} (A.14) \quad \|\partial_3 \mathfrak{Y}\|_{B^{0, \frac{1}{2}}} &\leq C \|\partial_3 \mathfrak{Y}\|_{\dot{B}^{\frac{1}{2}}} \leq C \|\partial_3 \mathfrak{Y}\|_{L^2}^{\frac{1}{2}} \|\nabla \partial_3 \mathfrak{Y}\|_{L^2}^{\frac{1}{2}} \\ &\leq C\varepsilon (\|\phi\|_{H^1} + \|\phi\|_{L^\infty_\nabla(H_h^1)}). \end{aligned}$$

On the other hand, it follows from a similar derivation of (A.8) that

$$(A.15) \quad \left\| \frac{\partial^3 w_3}{\partial z_i \partial z_k \partial z_\ell} \right\|_{L^\infty} \leq C\varepsilon \|\nabla \phi_3\|_{W^{2, \infty}}.$$

Then for  $\varepsilon \leq \varepsilon_3$  with  $\varepsilon_3$  being determined by

$$(A.16) \quad \varepsilon_3 \stackrel{\text{def}}{=} \min \left( \varepsilon_2, C^{-1} (1 + \|\nabla \phi_3\|_{W^{2, \infty}})^{-1}, C^{-1} (\|\phi\|_{H^3} + \|\phi\|_{L^\infty_\nabla(H_h^3)})^{-1} \right),$$

we get, by a similar derivation of (A.14), that

$$\|\nabla \mathfrak{Y}\|_{B^{1, \frac{1}{2}}} \leq C\varepsilon (\|\phi\|_{H^3} + \|\phi\|_{L^\infty_\nabla(H_h^3)}),$$

which together with (A.14) and (A.16) ensures (4.20).  $\square$

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